How to Win Nodes and Influence Networks: A Multidimensional Approach to Opinion Dynamics and Influence Games

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Abstract

In a period where socialization is hyper-reliant on digital platforms, permitting information to spread instantaneously, understanding strategies for optimal influence is vital. This paper develops a multidimensional, threshold-based opinion dynamics model– extending the work from DeGroot and Friedkin-Johnsen. Our model incorporates intertopic dependencies and external influence to model competitive diffusion over networks. We introduce a novel opinion update rule that incorporates local (neighbor) and global (external players) impact on opinion shifts. Through coupling linear threshold dynamics with traditional opinion models (FJ) and introducing intricacies of topic dependencies and multidimensional opinions, our model emulates realistic evolution of opinion and behavior. By simulating over synthetic and real-world data from the General Social Survey (GSS), we assess strategies of one and two-player models where influence is maximized. Results reveal that optimal strategies depend critically on initial opinion distribution, network topology, and the interdependence of topics. In particular, optimal strategies surface that leverage indirect influence by exploiting cross-topic relationships, and in the presence of competition, second movers gain a strategic edge. This work provides practical insight for designing self-regulating environments in polarized societies by strategically disseminating information. The implications of this research range from political campaigns, public health messaging, and ethical information diffusion.

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To my parents

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Chapter 1

Introduction

Motivations

How To Win Friends and Influence People by Dale Carnegie has been the prophecy on how individuals win friends and influence people for decades [9]. What about on a larger scale – how to win nodes and influence networks? Answering this question requires analysis on opinion dynamics and strategies for influence maximization. With individual interactions reaching unprecedented scales in light of technological innovation and social media dependence, connectivity is no longer restricted to physical distance causing opinion formulation to become more complex. Additionally, Bohn et al. estimate that individuals are exposed to more than 100,500 words corresponding to 34 gigabytes of information daily [6]. The surplus of data requires extensive need for decision on what information to accept– this often causes decision paralysis [2]. Thus knowledge of optimal strategies that exert one's influential dominance, persuading an adoption of ones stance, has attracted special attention [29][10][11][24][7][36][31].

Moreover, because opinions and behavior potentially threaten global issues (i.e. war, climate change, vaccinations, etc.) and impact essentially every system we attempt to measure (political, economic, social, etc.), comprehending what structures impact opinion dynamics is crucial. Furthermore, while true broadly but especially in the United States, polarization on issues has significantly increased– primarily with political opinions. Pew Research Data shows that on average, Democrats and Republicans are farther apart ideologically in the past three years than any time in the last 50 years [13][8]. With the Vanderbilt Unity Index (VUI)[37] predicting polarizing trends to continue, analyzing opinions shift that lead to extreme views is relevant to consider why this emerges and when this may actually be optimal.

1.1 Problem Statement

In an increasingly polarized and information-saturated society, individuals are constantly exposed to competing influences making opinion formation both complex and dynamic. While classical models like DeGroot[12] and Freidkin-Johnsen[16][15] capture internal opinion dynamics they fail to capture the role of external influences that strive to shape network-wide opinions. Furthermore, research that inspects optimal strategies for agents *outside* of the network seeking to maximize influence under game-theoretic frameworks is limited. Moreover, real-world opinions rarely exist in isolation or are independent and instead are often influenced by each other, yet existing models additionally lack this dimension. We argue that analyzing opinion dynamics by considering extensions to base models like DeGroot and FJ is foundational to understanding almost all systems (political, economic, social, etc.). Therefore, extending these models to represent information spread under current technological and social environments helps to accurately characterize systems that opinions and behavior impact. As such, we're seeking to answer the following questions:

1. What strategies are optimal for an external agent to maximize their influence over a population?

2. How does interdependence between topics affect optimal strategies?

3. Can we create self-regulatory environments by disseminating information in a strategic way?

Using complex networks through graphical methods to capture social dynamics, this research focuses on analyzing competitive diffusion over networks to understand how identity, behavior, and opinions evolve in response to external influences. By developing robust models that capture the dynamics of behavioral changes given an outside intervention, we offer insights into systems that exhibit variability and unpredictability due to the randomness of individual behaviors. This research explores both the dynamics of opinions and optimal strategies for a maximized global influence.

The paper is structured as follows. Chapter two discusses the existing models we use as a basis for our research and relevant literature that helps formulate our many extensions. Chapter three introduces the methodologies including the data used in conducting our research. Additionally, we define preliminary terms, variables, and notation necessary for understanding our model. Chapter four defines our model with only one external player and then extending to a two player model. Chapter five outlines numerical results under both our one-player and two-player models. Results are presented in the following steps: first under a small finite example using arbitrary parameters, then modeling over the General Social Survey dataset [20]. Finally, in chapter seven, we discuss the significance of our numerical results, providing conclusions on optimal strategies, opinion dynamics, and the structures that influence both. Here we further outline limitations, and future extensions to our research.

By bridging finite formulations theoretically and based on data, this research aspires to uncover principles that govern opinion dynamics and influence maximization in complex, interconnected systems. This research serves to benefit institutions, companies, and government entities that desire to exert maximum influence over populations with minimal effort. Applications of this include marketing, campaigning, and the establishment of a potentially self-regulating ecosystem in which the information-releasing agent selects the best initial node and message and as a result, the amount of effort needed to popularize an opinion is minimized. Think about wanting to encourage people to get vaccinated; is it possible for the population to come to that conclusion themselves by strategically releasing information? Or what about wanting to popularize a social or political movement? Running for president soon and want to gain popularity? This paper will offer insights to these types of questions. As an illustrative example, consider Bob trying to influence Alice and her friends to favor vanilla ice cream. Should Bob tell Alice directly, her best friend, or her parents and, what should Bob tell them?

Chapter 2

Opinion Dynamics Models

2.1 Literature Review

The study of influence in networks has evolved through multiple conceptual frameworks–from early models of information diffusion to sophisticated game-theoretic and multidimensional dynamics. To lay the foundation of our research, we review literature that has inspired our formulation of opinion dynamics and shifting behavior. Specifically investigating these dynamics under an environment with external agents aiming to maximize their influence without being embedded in the network structure themselves.

2.1.1 Theoretical and Conceptual Frameworks

Homophily has been a leading area of study in understanding how individuals within a network contribute to community structures. Specifically, homophily is the principle that breeds connections, supplying a theoretical ground for how influence is spread[26]. Researchers have stipulated that social influence and homophily often co-occur, making the separate study of them challenging[26]. Thus, a nuanced understanding of homophily is necessary for our research.

Khanam et al. discuss homophily and its implementation to predict link connections among individuals within a network [26]. The researchers introduce methods to compute links, referring to the Latent Dirichlet Model (LDA), which associates links of a document's topic distribution between people discussing related topics [26][5]. The study showed that connectivity among individuals is related to the distribution of topics [26]. This means that similarities in topics discussed or opinions on specific topics can be used to predict links between individuals. This type of prediction is useful when modeling social media networks where links may not be explicit[26]. To describe cascading dynamics, research[40][17] has introduced threshold models. These models require that for a non-adopting node to adopt new information, a fraction of their neighbors must exceed a global threshold[28][1]. Threshold models have been considered under static networks, and more recently, Min et al. have proposed co-evolving dynamics[30].

The theoretical foundations of homophily have motivated conceptions of threshold and cascade effects in networks. Information cascades outline a structure for how information is diffused, including mechanisms used to adopt information[40]. Much research has been conducted to model and learn structures of information cascades from large to granular networks[40]. Recent work has also focused on contextualizing this framework on modeling information spread on social media[40]. To analyze behavior on diffusion thresholds, López-Pintado [29] provides a dynamic sampling process that shows how visibility and information levels jointly determine whether behaviors become endemic or fade out.

Simultaneously, Borodin et al. extend threshold models to incorporate competitive diffusion[7]. The researchers use the well-known greedy $(1 - e^{-1})$ approximation for maximizing set functions[7]. They further show how competitive influence models are NP hard to achieve an approximation better than the square root of the optimal solution[7].

2.1.2 Structural and Stochastic Analyses of Diffusion

Parallel to theoretical frameworks of information diffusion, graph theorists have contributed to our understanding of network connectivity and its relational impact on diffusion. For instance, Erdos, Palmer, and Robinson[14] studied local connectivity thresholds in random graphs, establishing critical conditions under which influence can reliably propagate. Recently, researchers such such as Lelarge integrated cascade theories to analyze probabilistic models on random networks[28].

These probabilistic models laid the foundation for viewing networks as random bases for diffusion. This view is expanded on by Gyftopoulos et al.[18] in computational studies modeling DeGroot influence processes as Markov Decision Processes in order to verify influence strategies.

2.1.3 Foundational Models of Influence and Opinion Dynamics

Building on the theoretical understanding of influence in networks, numerous mathematical models have been developed to describe opinion dynamics. The DeGroot model [12] introduced a simple averaging process where agents iteratively adjust their opinions by taking a weighted average of neighboring views. While sleek, DeGroot assumes full convergence and thus is unrealistic to describe mechanisms for persistent disagreement. Further, this model assumes that individuals forego any initial opinion and simply update as the weighted average of their neighbors, presenting another oversimplification.

To address this, the Freidkin-Johnsen (FJ) model [16][15] introduces stubborn agents that retain bias toward their initial opinion. In doing so, equilibrium states with persistent disagreement are possible. The FJ model has become foundational for modeling opinion formation, providing realism, and analytical traceability.

2.1.4 External Sources and Game Theoretic Formulations

While early models assumed endogenous network influence, recent work considers agents seeking to influence networks as external entities. As a direct extension of threshold cascades, Jafari et al. extend linear threshold models to incorporate competitive diffusion [24]. They model individuals as heterogeneous and introduce optimal message selection in addition to node selection for influence maximization. A study by Out et al.[33], incorporates external biased media sources to the FJ model, exhibiting how persistent directional influence can be exerted. Similarly, Stella et al.[35] used mean-field game models to capture the interaction between stubborn agents and the general large population.

The strategic dimensions of influence have been explicitly formulated in a series of game-theoretic papers. Similar to[24], Irfan et al.[22] proposed a framework where two players compete to maximize influence by targeting specific nodes. Their research highlights optimal strategies' sensitivity to budget constraints, target selection, and network topology. Further, Jackson et al. examine how payoff structures and strategic complementarities affect equilibrium behavior in networks[23]. These models shift the focus from purely internal dynamics to the strategic role of external actors, an area that aligns directly with the current research.

2.1.5 Multidimensional and Interdependent Opinions

Real-world opinions are rarely uni-dimensional. Several recent contributions have tackled multidimensional opinion dynamics, where agents simultaneously hold beliefs over interrelated topics. Parsegov et al.[34] extended the FJ model to interdependent vector-valued opinions, incorporating cross-topic influences that reflect real-world belief systems in the multi-issue dependence structure (MiDS) matrix. Noipitak et al.[32] further explored such couplings through modified DeGroot-style models, demonstrating how structural interdependencies affect convergence. Such complexity also gives rise to cross-issue cascades, where influence on one dimension spills over into others, providing promising potential for indirect influence strategies.

2.1.6 Our Contributions

Though research has developed robust models to measure opinion dynamics under different settings, relatively few studies have explicitly optimized influence strategies for external agents in settings with:

- Internal agents holding multidimensional, interdependent opinions,
- Networks exhibiting threshold-based or nonlinear response behaviors,
- Indirect influence strategies,
- And external actors choosing when, where, and whom to influence.

This gap motivates the current research; to develop and analyze optimal strategies for external influencers seeking to maximize their impact over evolving opinion networks, under realistic constraints and agent behavior. Therefore in our research we develop a model with four key extensions to the traditional FJ [16] [15], DeGroot[12], and modified linear threshold models [24].

1. First, we extend our model to multiple dimensions, allowing opinions to be represented by L dimensional vectors as seen in section 3. This extension is similarly seen in the papers[34][32][19].
2. Secondly, rather than relying on their initial opinion at $t = 0$, which models stubbornness and retainment of one's initial opinions as framed in the FJ model[16][15], we use an individual's opinions at $t - 1$. By doing this, we

allow individuals' opinions to change more dynamically, arguing that individuals don't necessarily recall their opinions at conception but rather retain their opinions they held prior to an intervention that challenges them to change.

3. Third, by measuring an individual's opinion as the weighted average of their opinion at $t-1$, their neighbors, and an external influence, we're able to realistically model how an opinion is influenced by multiple sources (neighbors and influencing agents). This may be beneficial when inspecting if varying the weights contributes to the convergence of opinions

4. Fourth, motivated by Parsegrov et al. and their multiple influence decision matrix (MiDs)[34], we consider the interdependence of topics. Using statistical methods to measure the interdependence of topics, we propose a matrix, C, defined in section 3. This provides significant conclusions related to how optimal strategies may implore indirectly targeting an opinion via one highly related

Chapter 3

Methodology

To conduct our research and develop our extensions, we use the linear threshold model developed in[24] and FJ model[16][15] as base models for our extensions. In particular, we generalize the model in[24] to a multidimensional setting, enabling a more realistic representation of opinions across multiple interrelated topics, following the approach in[34]. We refine the update rule in Section 4.1 to reflect principles from the FJ model, allowing us to accurately model the neighboring influence on an individual's opinions. Topic interdependence is captured through a coupling matrix, similar to that used in[34].

Our extensions are structured around the model developed in[24] leveraging its inclusion of competitive external influence. Our multidimensional and competitive enhancements yield more realistic predictions for optimal strategies employed by external actors attempting to assert their influential dominance on a network. These extensions on this model offer realistic measures of optimal strategies for eternal influencers attempting to assert their influential dominance on a network.

We test our model through computational simulations using Python, first on synthetically generated data, and subsequently applying it to real-world data from the General Social Survey (GSS)[20]. This dual approach enables us to identify key diffusion patterns and validate our theoretical findings. In the following sections, we describe the methods of our modeling setup and initialization procedure.

3.1 Preliminaries and Notation

Lookup

For quick access, we define the terms and variables used in this research in Table 3.1

Preliminaries and Notation		
Term/Variable	**Notation**	**Definition**
Players	$P_k,\, k \in \mathbb{R}$	External players seeking to maximize influence over the population
Node/ Individuals	$N_i,\, i = 1, \ldots, N,\, N \in \mathbb{R}$	Agents par of the internal network
Graph/Network	$G = (N, E)$	N is the set of nodes, E is the set of edges
Topics	$L \in \mathbb{R}$	Number of topics measured (dimensionality of space)
Time	$t \geq 0,\ t \in \mathbb{R}$	Time variable
External Influence Weight	$\kappa \in [0, 1]$	Weight placed on influence of external agents
Neighbor Influence Weight	$\gamma \in [0, 1]$	Weight placed on influence of neighboring nodes
Message Propagation	$\delta \in [0, 1]$	Propagation parameter controlling to whom neighbor j sends a message
Sociability	$\beta \in [0, 1]$	Sociability parameter
Decay Rate	$\lambda \in [0, 1]$	Controls how quickly influence decays
Forwarding Threshold	$\theta^{\text{high}} \in [0, 1]$	Upper threshold for forwarding a message
Acceptance Threshold	$\theta^{\text{low}} \in [0, 1]$	Lower threshold for accepting a message
Topic Interdependence	$C \in \mathbb{R}^{L \times L}$	Interdependence matrix across the L topics
Neighbors of Node i	$N(i)$	Set of neighbors of node i
Neighbor j	$j \in N(i),\ j \neq i$	A neighbor j of node i
Link Connections	$w_{\text{link},i,j} \in [0, 1]$	Measures the level of connection between individual i and j
Normalization Factor	$Z_i = \sum_{j \in N(i)} w_{link,i,j}$	Normalization factor for weights from neighbors of node i
Alignment Score	$\text{Alignment}_{i,k}$	Measures alignment between node i and player k
Vector of Opinions	$A_i^l(t) \in \mathbb{R}^{L \times 1}$	Stores opinion $\alpha_i^l(t) \in [-1, 1]$ on each l topics for individual i
Vector of Messages	$T_k^l \in \mathbb{R}^{L \times 1}$	Stores messages on each $l \in [-1, 1]$ topic for player k

Table 3.1: Variables and notation used in our opinion dynamics model

3.2 Initial Setup

To construct our network, G we calculate the probability of connection by measuring the alignment between individuals' initial opinions, $A_i^l(t)$. We use this method of predictive link connectivity from theories of homophily[27][26] and research providing reasonable evidence to the validity of this prediction method[26][5]. Specifically, this is done using the kernel function:

$$\|A_i^l(t) - A_j^l(t)\| \tag{3.1}$$

Explicitly, we define the probability of a connection between two nodes (existence of an edge) as:

$$p_{i,j} = \exp\left(-\frac{1}{2\lambda}\|A_i^l(t) - A_j^l(t)\|^2\right) \tag{3.2}$$

where $\|.\|$ is the L_2 norm and λ as defined in Table 3.1.

The strength of the connection between nodes (the edge) is additionally dictated by the alignment of neighboring nodes' opinion vectors.

$$w_{\text{link},i,j} = f\left(\|A_i^l(t) - A_j^l(t)\|\right) \tag{3.3}$$

represents the strength of connection for neighboring nodes i and j and $f(\cdot)$ is a decreasing function. This is intuitive because as the distance between the opinions of two nodes increases, the connection between them decreases. The more aligned the nodes i and j are in opinions, the stronger $w_{\text{link},i,j}$. Explicitly we choose,

$$w_{\text{link},i,j} = \exp\left(-\|A_i^l(t) - A_j^l(t)\|\right) \tag{3.4}$$

To further initialize our network structure, we make the following assumptions:

Assumption 1: The network is undirected

Assumption 2: At time t, we observe the static opinion of each node i for all l topics, and the vector opinions, $A_i^l(t)$ is updated in discrete time steps.

Our set of players can be represented as $P_k = (P_1, P_2)$ for two external players, similar to Jafari's paper[24]. To model the competitive dynamics of these players we make additional assumptions:

Assumption 3: External Players P_k seek to maximize their influence over G (the network). All P_k are aware of each individual i's initial opinions on l topics.

Assumption 4: The players are seeking to influence the individuals' opinion on the same topic but in opposing ways

Assumption 5: Players can only target one topic and node i at $t = 1$

For Bob, this means he can only tell Alice, her friends, or her parents his message on favoring vanilla ice cream but cannot tell all of them at once. Additionally, Tom may want to influence Alice and her friends ice cream preferences as well, but instead of pushing a pro-vanilla agenda, Tom advocates for chocolate ice cream. Numerically Bob's push to vanilla can correlate to a push towards positive 1 on the topic of ice cream preference while Tom pushes to negative one for chocolate ice cream.

To optimize influence each player has two choices:

Choice 1: What message to send into the network

Choice 2: Where (what node) to being the initial diffusion of the message

3.2.1 Defining Parameters

To measure the initial opinions of individuals in the network and the magnitude of players messages on each l topic we define the structure of $A_i^l(t)$ and T_k^l [24]. At time $t = 1, 2, ..., T$

$$A_i^l(t) = [\alpha_i^1(t), \alpha_i^2(t), \alpha_i^3(t), ..., \alpha_i^L(T)] \tag{3.5}$$

represent the opinions of individual i on each l topic as an L-vector. Each $\alpha_i^l(t)$ is the individual opinion on topic l at time t and is $\in [-1, 1]$. Similarly,

$$T_k^l = [t_k^1, t_k^2, ..., t_k^L] \tag{3.6}$$

is and L vector of messages sent by each player. Each t_k^l represents the message from player k on a specific topic, l and is $\in [-1, 1]$.

Alignment$_{i,k}$ dictates how well aligned the opinion of node i is with player $k's$ message where

$$\text{Alignment}_{i,k} = \beta_k \cdot \big(1 - \frac{\|T_k^l - A_i^l(t)\|}{2\sqrt{L}}\big) \tag{3.7}$$

This controls the diffusion of a message where the more aligned $A_i^l(t)$ and T_k^l are, the more likely a message is propagated.

To demonstrate interdependence between topics we let

$$C = \begin{bmatrix} c_{l,l} & c_{l,l+1} & ...c_{l,L} \\ c_{l+1,l} & c_{l+1,l+1} & ...c_{l+1,L} \\ ... & & \end{bmatrix} \tag{3.8}$$

where

$$C \in \mathbb{R}^{L \times L}$$

Values $c_{l,l}$ represent a topic $l's$ influence on itself while, $c_{l,l+1}$ represents topic $l's$ influence on topic $l+1$. With C we're able to inspect how changing one opinion may affect shifts of other opinions and how that alters optimal strategies for influence.

Our construction of Alignment$_{i,k}$ is motivated by similar developments in Jafari's paper[24]. However, we adapt to a model that focuses on the node's alignment solely to the message rather than the players themselves[24]. We do this because under current social environments where information spread is rapid via technology the isolation for the source of a message has become challenging. Thus, the change of opinions have become primarily based on an individuals alignment with the message itself.

Additionally, the extension of existing opinion dynamic models[24][12][16] to multidimensional is inspired by Hazla and Parsegov's research[19][34] where similar approaches are taken. The definition of matrix C is further motivated by Parsegov's inclusion of a multi-issues dependence structure (MiDS) matrix[34]. These coupled extensions provide a more realistic approach to traditional opinion dynamic models. After developing our modeling algorithm in Sections 4.1 and 4.2, we run simulations when only 1 Player is present and then extend to 2 Players. In doing so, we're able to understand specifically how competition shifts optimal strategies. Additionally, we construct two different networks, G for arbitrary data and real-world data and keep G for each form of data consistent overall simulations. In doing so, we're able to keep node metrics consistent across all simulations (respective of the data being used) to analyze potential changes in node selection with and without competition present.

We run simulations in Python code using parallel processing to speed up run-time. In our code we let simulations continue until influence can no longer propagate, reflecting unrestricted time-steps. This offers important insight to measure potential convergence or level of effort exerted to influence a network.

3.3 Arbitrary Data

To measure the optimal strategies of players in our probabilistically connected network, G we first experiment with arbitrary data. Doing this gives us the basis for understanding notable trends that, if consistent with real data, present viable conclusions.

Table 3.2: Initialized parameters for our model used for all arbitrary simulations

Parameter	**Value**
Number of nodes (N)	50
Number of topics (L)	3
Socialization parameter (β_i)	1
High threshold (θ^{high})	0.7
Low threshold (θ^{low})	0.3
Propagation threshold (δ)	0.5
Influence decay parameter (λ)	0.5

Table 3.2 depicts how the parameters are initialized. We choose $A_i^l(t)$ for each i randomly from a uniform distribution for values between [-1,1]. For the Players we choose T_k^l in the following steps:

1. Choose the topic the Player is targeting
2. Send a polarizing message in the index of T_k^l that corresponds to the topic chosen in step 1

For example if we consider three topics, $L = 3$ and topic 1 represents ice cream preference, Bob influencing towards a vanilla preference would send the message:

$$T_1^{L=3} = [1, 0, 0] \tag{3.9}$$

If Tom decides to challenge Bob, advocating for a chocolate ice cream preference then following **Assumption 4**, Tom sends the message:

$$T_2^{L=3} = [-1, 0, 0] \tag{3.10}$$

3.4 GSS Data

To validate the trends observed from simulations using arbitrary data, we test our model using the General Social Survey (GSS)[20], specifically using longitudinal data. GSS is a national survey that measures the attitudes and behaviors of adults in the US taking place annually or biannually since 1972 [20]. We extract data from the GSS longitudinal panel study from 2018-2020. In this study, the same adults are surveyed once in 2018 and then again in 2020 on the same/similar questions from the initial survey in 2018. This measures changes in attitudes and behavior over two years, providing use when measuring how changes in opinions on different topics move together. The initial dataset collects data on $N = 2,347$ participants.

The surveys are administered either over the phone or through an online platform and are offered in both English and Spanish. Along with individuals' opinions on topics ranging from same-sex marriage to views on federal funding, the dataset provides individuals' demographic information. This can be useful in analyzing whether there are significant trends among demographic groups pertaining to influence receptivity.

For the purposes of this research, we focus on three topics present in the dataset: affirmative action, gun permits, and political party affiliation. These topics are measured in the survey with raw values shown in Table 3.3.

We clean the dataset by dropping individuals with `NA` values for any of these topics in addition to ones measuring demographic features for future research. We remove individuals who responded to Political Party with a raw value of 7 (Other party) because we'd like to confine results to minimal ambiguity in opinions. To ensure opinions remain $\in [-1, 1]$ we normalize the raw values which is also shown in Table 3.3:

Table 3.3: General Social Survey (GSS) data used in our research with original values from the dataset normalized and the mappings of what the values mean

Topic	Raw Value	Normalized Value	Meaning
1: Affirmative Action	1	1.00	Strongly favors
	2	0.33	Not strongly favors
	3	-0.33	Not strongly opposes
	4	-1.00	Strongly opposes
2: Gun Permits	1	1.00	Favor
	2	-1.00	Oppose
3: Political Party	0	1.00	Strong Democrat
	1	0.67	Not very strong Democrat
	2	0.33	Independent, close to Democrat
	3	0.00	Independent (neither, no response)
	4	-0.33	Independent, close to Republican
	5	-0.67	Not very strong Republican
	6	-1.00	Strong Republican

We chose these topics because we believe they have a strong enough relationship that allows us to measure their interdependence for our construction of C. To construct C we calculated the covariance between topics from our cleaned data using opinions captured in 2018 and then again in 2020. The resulting matrix with $i = j = 1$ = Affirmative Action, $i = j = 2$ = Gun Permits, and $i = j = 3$ = Political

Party is:

$$\begin{bmatrix} 0.92 & 0.02 & 0.06 \\ 0.01 & 0.94 & 0.05 \\ 0.17 & 0.23 & 0.62 \end{bmatrix} \tag{3.11}$$

Our final dataset consists of $N = 282$ individuals that are represented as nodes in our network. This decreased value of N is a result of additionally cleaning for the demographic information.

yearid	**affrmact_1b**	**gunlaw_1b**	**partyid_1b**	**affrmact_2**	**gunlaw_2**	**partyid_2**
20180001	-1.00	1.00	-0.67	-1.00	1.00	-1.00
20180006	0.33	1.00	0.33	-0.33	1.00	0.33
20180011	-1.00	1.00	-1.00	-1.00	1.00	-1.00
20180016	-0.33	1.00	0.00	-1.00	1.00	0.00
20180063	0.33	1.00	1.00	1.00	1.00	1.00

Table 3.4: First five entries of the GSS dataset cleaned and normalized for values $\in [-1, 1]$

First five entries of the GSS dataset cleaned and normalized for values $\in [-1, 1]$. yearid is the unique code that tracks an individual. affrmact_1b, gunlaw_1b, and partyid_1b are the opinions on Affirmative Action, Gun Permits, and Political Party in 2018. affrmact_2, gunlaw_2, and partyid_2 are the opinions on Affirmative Action, Gun Permits, and Political Party in 2020.

Using the opinions of each individual, we construct their $A_i^l(t)$ opinion vector where

$$\text{Topic 1} \rightarrow \text{Affirmative Action}$$
$$\text{Topic 2} \rightarrow \text{Gun Permits}$$
$$\text{Topic 3} \rightarrow \text{Political Party}$$

Before running simulations on the data, we inspect the initial alignment of the population as it may describe our results and be important for optimal strategy considerations. To gauge this, we sum the amount of individuals who share the same initial opinion for each topic. Results show that the initial network is calibrated in the following way:

1. **Topic 1 (Affirmative Action):** Majority of the population aligned to -1, meaning they are against Affirmative Action

2. **Topic 2 (Gun Permits):** Majority of the population aligned to +1, meaning they favor Gun Permits

3. **Topic 3 (Political Party):** Majority of the population aligned to 0/+1, meaning they are Independent, leaning Democratic

With arbitrary and real data[20] we can compare consistent trends that invoke significant insight. Specifically it offers robust knowledge on how dynamics are impacted by external interventions, what strategies are optimal for influence maximization, and what characteristics of a network impact both and are therefore important to consider. The full explanation of our modeling algorithm and results from simulations are presented in the preceding chapter.

Chapter 4

The Model

To define our model as mentioned in Section 3.2.1, the diffusion of T_k^l and the spread of $P_k^{\prime}s$ influence depends on how well aligned T_k^l is with the opinions of an individual, $A_i^l(t)$ at the time of diffusion. We formally define the **Acceptance** and **Forwarding** algorithms below.

4.1 Acceptance Algorithm

1. If

$$\text{Alignment}_{i,k} > \theta^{low} \tag{4.1}$$

node i accepts player $k's$ message (T_k^l) and becomes active. Strong alignment increases the probability that a message is accepted. From **Assumption 5** at $t = 1$ only one node can be activated, forcing Players to optimize for the best initial node to cascade influence.

Given node i accepts and is activated

$$A_i^l(t+1) = (1-\kappa-\gamma)A_i^l(t) + C\kappa T_k^l + \gamma \sum_{j\in N(i)} \frac{w_{link,i,j}}{Z_i} A_j^l(t) \tag{4.2}$$

represents how $i's$ opinions, $A_i^l(t)$, are updated with κ and γ representing the weight of external influence and neighbor influence respectively 3.1. We can also represent Equation 4.2 in matrix form:

$$A_i^l(t+1) = DA_i^l(t) + C\kappa T_k^l, \tag{4.3}$$

where D is the weighted adjacency matrix and:

$$D_{ij} = \begin{cases} (1-\kappa-\gamma) & \text{if } i = j, \\ \gamma \frac{w_{link,i,j}}{Z_i} & \text{if } i \neq j. \end{cases} \tag{4.4}$$

This matrix D is also row-stochastic.

Definition 4.1.1: A matrix D is said to be row-stochastic if:

- All elements are non-negative: $D_{ij} \geq 0$.
- Each row sums to 1:

$$\sum_j D_{ij} = 1, \quad \forall i.$$

Proof. of the row-stochasticity of matrix D:

To verify if our matrix D is row-stochastic, we sum its entries row-wise:

$$\sum_j D_{ij} = (1 - \kappa - \gamma) + \gamma \sum_{j \in N(i)} \frac{w_{link,i,j}}{Z_i}. \tag{4.5}$$

Since $Z_i = \sum_{j \in N(i)} w_{link,i,j}$, it follows that:

$$\sum_{j \in N(i)} \frac{w_{link,i,j}}{Z_i} = 1. \tag{4.6}$$

Thus:

$$\sum_j D_{ij} = (1 - \kappa - \gamma) + \gamma \cdot 1 = 1. \tag{4.7}$$

$\square$

Since all elements of D are non-negative, this confirms that D is row-stochastic by Definition 4.1. This property guarantees that opinion propagation follows a probabilistic interpretation and contributes to potential formulations of stability of the opinion formation process [34].

4.2 Forwarding Algorithm

After node i accepts and becomes active, it then decides whether to forward player $k's$ message to its neighbors $j \in N(i)$.

2. If

$$\text{Alignment}_{i,k} > \theta^{high} \tag{4.8}$$

node i decides to forward the message T_k^l but only to neighbors j, $\forall j \in N(i)$ that satisfy:

$$w_{\text{link},i,j} > \delta \tag{4.9}$$

This means, if the link strength calculated by Equation 3.3 is strong enough for interactions to occur, the message is diffused into node j. The diffusion of the message continues at $t \geq 2$ following the **Acceptance Algorithm** and **Forwarding Algorithm**.

The structure of these algorithms reflects the realistic behavior of homophily. Individuals don't readily interact (and forward information) unless they share similar beliefs [27][26]. This dynamic describes much of what models like Hugh's paper discusses occurs in our natural world [21].

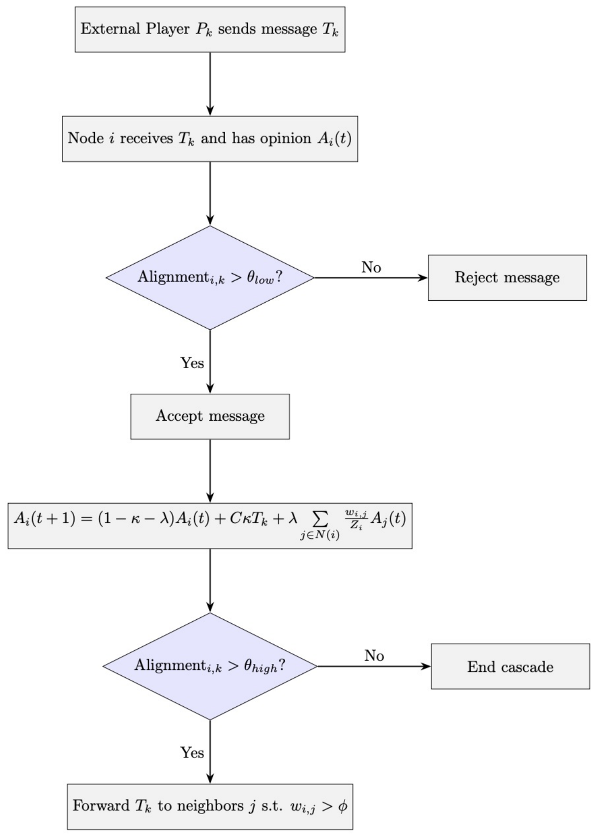


Figure 4.1: Flowchart of the opinion update process.

Flowchart of the opinion update process. Node i accepts a message T_k from external player P_k if the alignment exceeds the threshold θ_{low}, then updates its opinion using a weighted combination of its past opinion, the message, and neighbors' views. If alignment exceeds θ_{high}, the node forwards the message to neighbors with sufficient link strength $w_{i,j}$.

Further, individuals' own opinions reflect a combination of their neighbors' views while maintaining their original opinions[16][15]. Friedkin Johnsen's (FJ) model captures this, extending the DeGroot model, which only represents an individual's opinions as the weighted average of their neighbors [12][16][15]. Neither model, however, considers the impact of external influence on an individual's opinion, thus, our model again provides a key extension.

4.2.1 One Player

To capture the effects of this external influence on opinions, we first consider only one external Player. The external Player maximizes influence on the target topic by finding the optimal node and message at $t = 1$. We define the Player's payoff as

$$J_k = \sum_i \left[\alpha_i^{l^*}(T) - \alpha_i^{l^*}(0)\right] \tag{4.10}$$

which is the sum of each individual's opinion shift toward the Players' direction of influence. Here, l^* represents the targeted topic, and T is the last time step, when influence no longer propagates and diffusion ends.

The optimization problem for an optimal node i and message T_k^l that maximizes a Player's payoff in the +1 influence direction is:

$$\max_{i^*, T_k^{l^*}} J_k = \sum_i \left[\alpha_i^{l^*}(T) - \alpha_i^{l^*}(0)\right] \tag{4.11}$$

subject to:

$$t_k^l \in [-1, 1] \tag{4.12}$$

Where, $i^*, T_k^{l^*}$ are the optimal choices of node and message respectively. When the

Player's influence direction is -1, the objective shifts to:

$$\min_{i_2^*, T_2^{l^*}} J_2 = \sum_i \left[\alpha_i^{l^*}(T) - \alpha_i^{l^*}(0) \right] \tag{4.13}$$

with the same constraints 4.12 but accounting for the negative direction.

The algorithm for simulating the 1-Player model is below, and the results are discussed in Chapter 5:

1. **Initialization**
   (a) Initialize all the parameters
   (b) Populate opinions $A_i^l \in [-1, 1]$ for each node i and each topic l.
   (c) Construct G connecting nodes if $p_{i,j} >$ a random number from a uniform distribution $\in [0, 1]$.
   (d) Compute initial link weights w_{ij} for all edges.
2. **Optimize Initial Strategy**
   (a) Solve for the optimal node i^* and message vector T_k^{l*} that maximize the total influence on the target topic l^*.
   (b) Let (i^*, T_k^{l*}) be the optimal result.
3. **Simulate Opinion Diffusion (for $t = 0, 1, \ldots, T-1$)**
   (a) **At $t = 1$:**
      i. Compute the alignment of i^* with T_k^{l*}.
      ii. If i^* accepts the message based on θ^{low}:
         A. Update opinions: $A_i^l \leftarrow DA_i^l + C\kappa T_k^{l*}$.
         B. Recalculate link weights for i^*.
         C. Check if i^* forwards the message based on θ^{high}.

D. For each neighbor j of i^*, forward the message to j if $w_{ij} > \delta$.

(b) **For** $t \geq 2$**:**

i. For each active node i:

A. For each neighbor j:

B. Compute alignment of j with T_k^{l*}.

C. If j accepts the message based on θ^{low}:

D. Update opinions: $A_j^l \leftarrow DA_j^l + C\kappa T_k^{l*}$.

E. Recalculate link weights for j.

F. Check if j forwards the message based on θ^{high}.

G. For each neighbor j' of j, forward the message to j' if $w_{jj'} \geq \delta$.

(c) Continue diffusion from (b). Stop if no new nodes accept or forward the message.

4. **Compute Total Influence**

(a) Sum the change in opinions on the target topic l^* across all nodes.

5. **Return Results**

(a) Return total influence, optimal node i^*, and optimal message vector T_k^{l*}.

4.2.2 Two Player

Extending our model to include competitive diffusion, we consider a system with 2 external Players that mimics a Stackelberg game [38]. We fix the influence directions of each Player to reflect **Assumption 4** with Player 1 pushing to +1 and Player 2 pushing to -1.

Initially, all nodes are inactive, and Players can each only activate one node at $t = 1$ (**Assumption 5**). When node i is contacted by only one of the two Players, the decision to accept follows from Section 4.1. However, if i receives a message from both Players and the acceptance threshold is met, i

1. Computes the $Alignment_{i,k}$ for each k Player, $k = \{1, 2\}$
2. Accepts the message of player k who has a higher $Alignment_{i,k}$ (more aligned)

The decision to forward remains the same from Section 4.2.

Players take turns moving first, and the second mover optimizes their strategy in response to the observed fixed strategy of the opposing Player. The optimization problem for Player 1 and Player 2 respectively is:

$$\max_{i_1^*, T_1^{l^*}} J_1 = \sum_i \left[\alpha_i^{l^*}(T) - \alpha_i^{l^*}(0)\right] \tag{4.14}$$

subject to:

$$t_k^l \in [-1, 1] \tag{4.15}$$

$$\min_{i_2^*, T_2^{l^*}} J_2 = \sum_i \left[\alpha_i^{l^*}(T) - \alpha_i^{l^*}(0)\right] \tag{4.16}$$

subject to:

$$t_k^l \in [-1, 1] \tag{4.17}$$

The relationship between the payoffs of Player 1 and Player 2 is:

$$J_2 = -J_1 \tag{4.18}$$

This model effectively depicts the dynamics of competition prevalent in society.

For example, Bob and Tom are vying to maximize their influence on Alice and her friends' ice cream preferences in opposing directions. Bob pushes to positive 1 reflecting a vanilla preference while Tom pushes towards -1 for chocolate.

The algorithm for our 2-Player model is below, with the results discussed in the following chapter.

1. **Initialization**
   (a) Initialize all parameters
   (b) Populate initial opinions, $A_i^l \in [-1, 1]$ for each node i and each topic l.
   (c) Construct G connecting nodes if $p_{i,j} >$ a random number from a uniform distribution $\in [0, 1]$.
   (d) Compute initial link weights w_{ij} for all edges.
2. **Optimize Initial Strategies**
   (a) Fix P_1's initial node and message. Solve for P_2's optimal node i_2^* and message vector T_2^{l*} to optimize J_2.
   (b) Fix P_2's initial node and message. Solve for P_1's optimal node i_1^* and message vector T_1^{l*} to optimize J_1.
3. **Simulate Opinion Diffusion (for $t = 0, 1, \ldots, T-1$)**
   (a) **At $t = 1$:**
      i. Players P_1 and P_2 send their messages T_1^{l*} and T_2^{l*} to their chosen nodes i_1^* and i_2^*.
      ii. For each player P_k:
         A. Compute alignment of i_k^* with T_k^{l*}.
         B. If i_k^* accepts the message based on θ^{low}:
         C. Update opinions: $A_{i_k^*}^l \leftarrow DA_{i_k^*}^l + C\kappa T_k^{l*}$.
         D. Recalculate link weights for i_k^*.
         E. Check if i_k^* forwards the message based on θ^{high}.

F. For each neighbor j of i_k^*, forward the message if $w_{i_k^* j} \geq \delta$.

iii. Resolve conflicts if both players target the same node by choosing the message with higher alignment or randomly if alignments are equal.

(b) **For $t \geq 2$:**

i. For each active node i:

A. For each neighbor j:

B. Compute alignment of j with messages T_1^* and T_2^*.

C. If j receives and accepts only one message based on θ^{low}:

D. Update opinions:$A_j^l \leftarrow DA_j^l + C\kappa T_k^{l*}$.

E. If j receives both messages and accepts both:

F. Resolve conflict by choosing the message with higher alignment or randomly if alignments are equal.

G. Update opinions of j accordingly.

H. Recalculate link weights for j.

I. Check if j forwards the accepted message(s) based on θ^{high}.

J. For each neighbor j' of j, forward the message if $w_{jj'} \geq \delta$.

(c) Continue diffusion at (b). Stop if no new nodes accept or forward the message.

4. **Compute Payoffs**

(a) Calculate J_1 and $J_2 = -J_1$ summing the change of l^* across all nodes and correcting J_2 for influencing in the negative direction.

5. **Return Results**

(a) Return payoffs J_1, J_2, and optimal strategies (i_1^*, T_1^*), (i_2^*, T_2^*).

Chapter 5

Numerical Results

In this chapter, we present the results of our simulations conducted under both the 1-Player and 2-Player models. Starting with arbitrary data in the 1-Player model, we progressively expand to the 2-Player model, utilizing the General Social Survey (GSS) data[20]. We will walk through these results step-by-step, highlighting the effects on individual behavior and strategic trends of Players for influence maximization.

5.1 Arbitrary Data

Using the same graph, G, for our arbitrary simulations, the average node metrics are:

- **Average Degree** = 21.28
- **Average Clustering Coeff.** = 0.52
- **Average Degree Centrality** = 0.43
- **Average Betweenness Centrality** = 0.0290

5.1.1 One Player

Using the initial parameters as seen in Table 3.2 without considering the interdependency of topics, meaning Equation (4.2) doesn't account for matrix C, we run 50 simulations and observe the following results:

Table 5.1: Simple Finite Results (1 Player Push to 1).

Topic	**Max Total Influence**	**Optimal Node**	**Optimal Message**
1: Affirmative Action	17.22	14	[0.81 -0.24 0.71]
2: Gun Permits	16.46	17	[0.00, 1.00, 0.00]
3: Political Party	16.45	42	[0.00, 0.00, 1.00]

Table 5.2: Simple Finite Results (1 Player Push to -1).

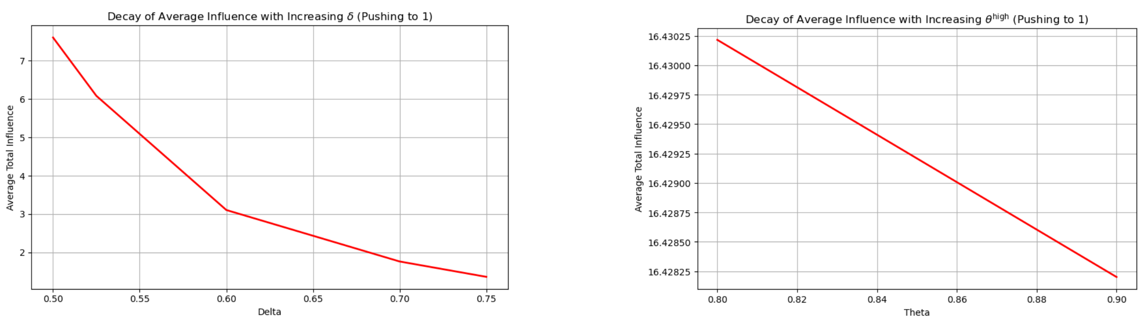
Topic	**Max Total Influence**	**Optimal Node**	**Optimal Message**
1: Affirmative Action	14.04	23	[-0.99, -0.66, -0.81]
2: Gun Permits	13.54	46	[0.00, -1.00, 0.00]
3: Political Party	15.54	3	[-0.54, -0.08, -0.97]

In both scenarios, all $N_i = 50$ individuals in the undirected network are initialized with parameters $\beta = 1, \theta^{low} = 0.3, \theta^{high} = 0.7, \delta = 0.5$ and connected via 3.2 with $\lambda = 0.5$

In the absence of competition and without considering the interdependence of topics, the optimal strategy for a Player is to send a polarizing message targeting the topic of interest. For instance, when aiming to push the opinion toward 1, the optimal message for the target topic is one that is close to 1 (or strongly positive).

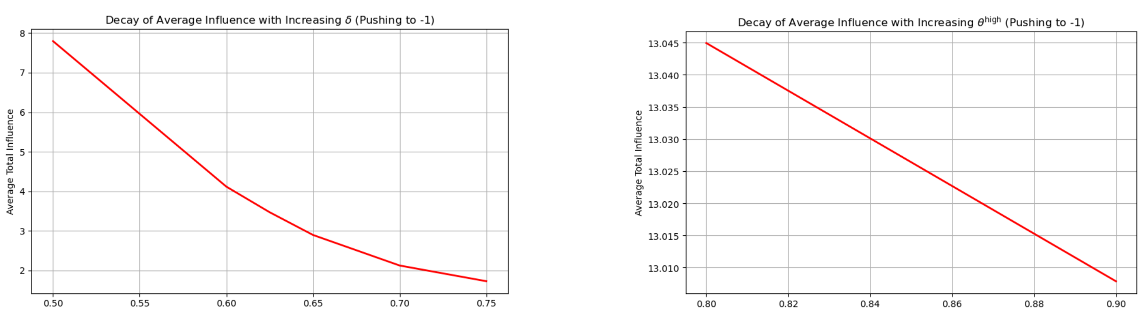
However, the results also suggest that a mixed message strategy, where the message does not focus solely on the target topic, yields the highest overall influence. This implies that while it's essential for the influencing agent to send a message that is strongly aligned with their desired influence direction for the target topic, it is equally important not to overly concentrate the message on the target topic itself.

To evaluate our model's parameter sensitivity on influence, we examine the effects of incrementally increasing θ^{high} and δ. We focus on θ^{high} and δ since they dictate the propagation and cascade of information. We increase these incrementally by a factor of 0.1. As expected, these increases resulted in lower average influence in both 1-Player scenarios. Notably, δ is presented to be more sensitive, causing a lower diffusion of influence than θ^{high}. This implies that the alignment between individuals plays a more critical role in influence than an individual's alignment with the message itself.



(a) Increasing δ by 0.1 (b) Increasing θ^{high} by 0.1

Figure 5.1: Incrementally increasing parameters when pushing to 1



(a) Increasing δ by 0.1 (b) Increasing θ^{high} by 0.1

Figure 5.2: Incrementally increasing parameters when pushing to -1

Thus, an optimal influencing agent strategy involves:

1. Prioritizing messages that are strongly aligned with the desired direction for the target topic
2. Ensuring that the message is not exclusively focused on the target topic but rather considers a more balanced approach that allows influence to extend to related topics
3. Focusing on inter-nodal alignment rather than solely aligning the message with the optimal node

5.1.2 Two Player

To inspect how competition impacts optimal strategies, we extend our arbitrary data simulations to our two-player model. Running 50 simulations under the same assumptions and parameterizations, we observe:

Table 5.3: Arbitrary Data (2 Player, Player 1 Goes First).

Topic	**Player 1**	**Player 2**
1: Affirmative Action	**Max Influence:** 3.83 **Optimal Node:** 17 **Message:** [-0.15, -0.08, 0.57]	**Max Influence:** 9.17 **Optimal Node:** 1 **Message:** [-1.00, 0.00, 0.00]
2: Gun Permits	**Max Influence:** 2.17 **Optimal Node:** 3 **Message:** [0.33, -0.38, 0.15]	**Max Influence:** 6.08 **Optimal Node:** 49 **Message:** [0.00, -1.00, 0.00]
3: Political Party	**Max Influence:** 3.07 **Optimal Node:** 42 **Message:** [0.09, 0.09, -0.54]	**Max Influence:** 10.54 **Optimal Node:** 3 **Message:**[-0.52, -0.70, -0.76]

Table 5.4: Arbitrary Data (2 Player, Player 2 Goes First)

Topic	**Player 1**	**Player 2**
1: Affirmative Action	**Max Influence:** 8.27 **Optimal Node:** 11 **Message:** [0.47, -0.11, 0.75]	**Max Influence:** 3.31 **Optimal Node:** 1 **Message:** [0.00, 0.00, -1.00]
2: Gun Permits	**Max Influence:** 6.23 **Optimal Node:** 17 **Message:** [-0.39, 0.29, 0.52]	**Max Influence:** 4.86 **Optimal Node:** 20 **Message:** [0.02, 0.12, 0.11]
3: Political Party	**Max Influence:** 6.37 **Optimal Node:** 42 **Message:**[0.23, 0.26, 0.91]	**Max Influence:** 5.16 **Optimal Node:** 33 **Message:** [-0.21, 0.15, 0.21]

In both 2-Player scenarios, all $N_i = 50$ individuals in the undirected network are initialized with parameters $\beta = 1, \theta^{low} = 0.3, \theta^{high} = 0.7, \delta = 0.5$ and connected via 3.2 with $\lambda = 0.5$

We find that the player who moves second generally achieves higher influence across all topics. This outcome can be attributed to the second player's ability to observe the first player's strategy and then respond optimally. By adapting their strategy based on the first player's actions, the second player can capitalize on the information already provided and make more effective moves, resulting in a higher total influence. Another consistent finding is that mixed message strategies tend to lead to higher influence. Suggesting that the threshold dynamics from our model impact players' strategies. This means that while targeting a topic, Players attempt to produce a T_k^L that is most optimal in alignment with initial opinions for a broader cascade effect.

5.2 GSS Data

To test if the results are consistent with real-world data, we use the GSS longitudinal survey[20]. Maintaining the above assumptions, we run simulations in the absence of the interdependence matrix. Then, to inspect how dependence between topics influences optimal strategies, we consider the C matrix as in Equation 4.2. Before beginning simulations, we describe key data from our network.

Table 5.5: Average Connection Probability Per Topic

Topic	**Average Probability**
affirmative_action	0.731862
gun_laws	0.670030
party_id	0.719685

The average probability per topic is measured by iterating over each pair of N_i and computing the alignment via 3.1 on each distinct topic separately. Then, using 3.2 for the probability of connection for each topic, we average the results with the values presented in the table above.

Calculating the average connection probability per topic reveals which topic the majority of the population is most collectively aligned with, offering key insight into network connectivity and potential cascade dynamics. that opinions on Affirmative Action are most likely to create connections between individuals, followed closely by opinions on Political Party affiliation. Notably, opinions on Gun Permits are less likely to result in connections compared to the other topics. This suggests that topics related to race and political identity, such as Affirmative Action and Political Party affiliation, tend to foster greater opinion alignment and social connection compared to topics like Gun Permits, which may be more divisive.

The higher connection probabilities for Affirmative Action and Political Party affiliation hint at the lower stubbornness of these topics, indicating that opinions on these issues are more easily swayed or aligned within the network. In contrast, the lower connection probability for Gun Permits suggests that opinions on this topic are more polarized and potentially more resistant to change, making it a stubborn topic that may require more targeted influence strategies.

These findings raise intriguing questions about the relationship between race-based topics (i.e. Affirmative Action) and non-race-based topics (i.e. Gun Permits) in fostering connections and shaping influence within a network. It appears that race-related topics might more readily unite individuals, while non-race-related issues like Gun Permits could require more effort to bridge divides. This insight will be valuable as we proceed with simulating influence strategies on these topics, as understanding the stubbornness of topics can guide more effective targeting and message crafting in our simulations.

Additionally, using the same G for our GSS data simulations, the average node metrics are:

- **Average Degree** = 101.00
- **Average Clustering Coeff.** = 0.54
- **Average Degree Centrality** = 0.36
- **Average Betweenness Centrality** = 0.0023

5.2.1 One Player

Using the probabilistically connected network and the initialization we had for the thresholds and sociability parameters from Table 3.2, we run the 50 for both scenarios below.

1. **Scenario 1:** Outside player wants to influence opinions toward 1

2. **Scenario 2:** Outside player wants to influence opinions toward -1

Further, we consider each scenario under the environment where topics are treated independently when their interdependence is considered. Doing this provides evidence as to how topic dependence shifts optimal strategies and when it's leveraged.

Table 5.6: Finite Results GSS Data (1 Player Push to 1)

Topic	Without C	With C
1: Affirmative Action	**Max Influence:** 111.10 **Optimal Node:** 70 **Message:** [0.92, 0.85, -0.11]	**Max Influence:** 108.67 **Optimal Node:** 136 **Message:**[0.96, 0.58, -0.41]
2: Gun Permits	**Max Influence:** 35.73 **Optimal Node:** 276 **Message:** [0.12, 0.97, -0.28]	**Max Influence:** 36.43 **Optimal Node:** 253 **Message:** [0.13, 0.97, -0.31]
3: Political Party	**Max Influence:** 68.23 **Optimal Node:** 259 **Message:** [0.57, 0.89, 0.98]	**Max Influence:** 62.99 **Optimal Node:** 106 [0.42, 0.94, 0.86]

Table 5.7: Finite Results GSS Data (1 Player Push to -1)

Topic	Without C	With C
1: Affirmative Action	**Max Influence:** 24.28 **Optimal Node:** 126 **Message:** [-0.98, -0.58, -0.56]	**Max Influence:** 20.89 **Optimal Node:** 126 **Message:**[-0.94, -0.37, -0.45]
2: Gun Permits	**Max Influence:** 120.25 **Optimal Node:** 66 **Message:** [-0.13, -0.69, 0.39]	**Max Influence:** 118.80 **Optimal Node:** 149 **Message:** [-0.82, -0.54, 0.64]
3: Political Party	**Max Influence:** 65.78 **Optimal Node:** 70 **Message:** [-0.19, 0.94, -0.99]	**Max Influence:** 55.25 **Optimal Node:** 200 [-0.94, 0.36, -0.90]

Maximal Influence

When topics were treated as independent, the maximum level of influence was generally higher across all topics. However, when intertopic dependencies were considered and the Player pushed to +1, Topic 2 (Gun Permits) saw a slight increase in influence. These shifts imply that the interdependence of topics adds complexity to the selection of optimal nodes and message vectors. This indicated that with dependencies, influence strategies become more nuanced.

Optimal Topic:

- Topic 1 (Affirmative Action) was most influenced when the Player pushed the opinion toward 1. This result makes sense, given that the network's initial status quo was aligned with a -1 view on Topic 1, leaving more room for influence in the opposite direction.

- Topic 2 (Gun Permits) was most influenced when the Player pushed the opinion toward -1. again, reflecting that pushing in the opposite direction of initial alignment offered more room for influence.

- Topic 3 (Political Party) showed a more balanced influence with relatively similar levels of influence in either influence direction, likely due to a more moderate initial alignment of the networks' opinions.

Optimal Node

Indeed, when pushing to +1 or -1, the optimal node selection shifted with interdependencies considered. Although pushing to +1 was still most influential on Topic 1 (Affirmative Action) in both cases, when we included dependencies, the Player opted for nodes with higher degrees. The same is true for when the Player pushed to -1 and Topic 2 (Gun Permits). On topics where the Player had less influence, respective of their push direction, the Player focused on selecting a node with a higher clustering coefficient. This indicated that when influence is harder to achieve, a Player prioritizes nodes with stronger local connections within their community. Beyond this, in both directions of influence, the optimal nodes shifted to those more aligned with the networks' initial views in a particularly polarizing way. Specifically, optimal nodes became ones with views of -1 for Affirmative Action and 1 for Gun Permits, matching the initial status quo. This implies that with intertopic dependencies, optimal nodes are those that can leverage existing polarization to drive influence.

Optimal Message

The presence of dependencies provided fascinating insights on the optimal message. When targeting a topic in the same direction that the initial networks' opinions agree with, Players sent a message in the opposite direction for a topic related to the targeted one. This may initially seem to be a response to threshold requirements, but it goes deeper. By sending an opposite message on a related topic, the Player pretends to align with the opposing individual on the related topic. This effectively persuades individuals to accept their influence direction on the target topic. The Player leverages indirect influence, subtly manipulating individuals by aligning themselves on related topics. Further on topics that the initial networks opinions were moderate, the optimal strategy for a Player was to increase the existing polarity on topics that agreed with the Players influence direction. This suggests that capitalizing on existing polarization helps persuade moderate opinions to your view. The message strategy then indicates:

- Polarize topics where the network is already polarized (i.e. Topic 1 and Topic 2)
- Moderate the message for topics where there is potential to sway opinions in the opposite direction
- Align with the status quo on topics that are already polarized and use these to influence a topic that is not yet polarized

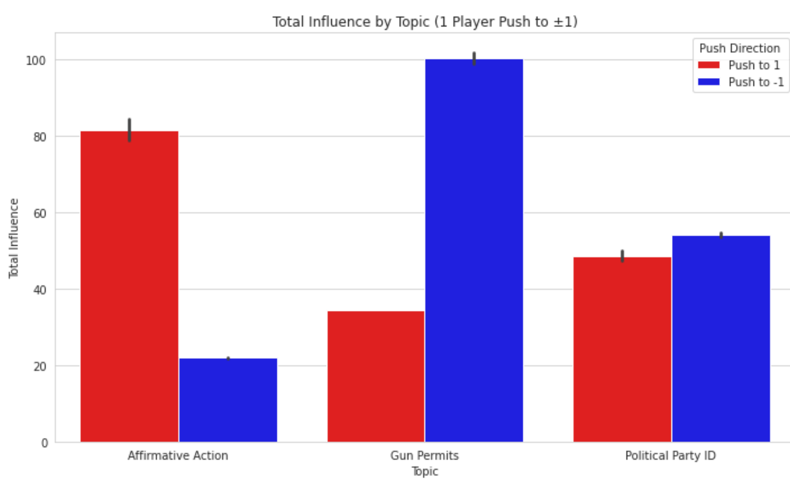


Figure 5.3: Average Influence Over All Topics for Both 1-Player Scenarios Without C, Using GSS Data

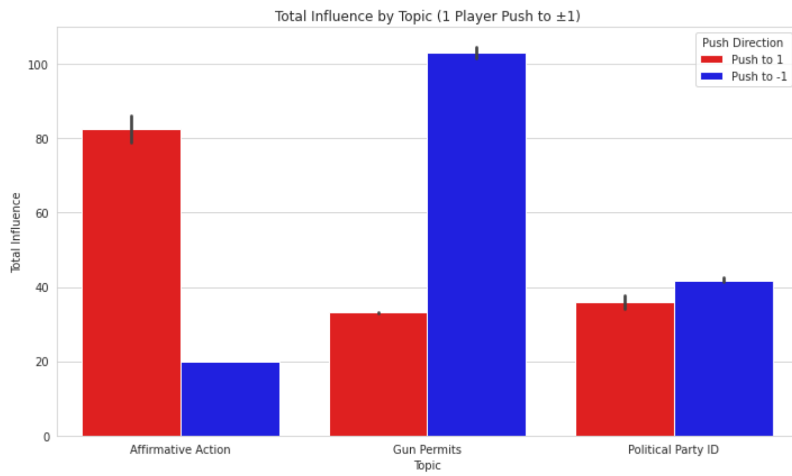


Figure 5.4: Average Influence Over All Topics for Both 1-Player Scenarios With and With C, Using GSS Data

The average influence shows that pushing towards 1 generally leads to better influence maximization across all topics. Notably, it achieves higher influence on its least optimal topic (Topic 2) than pushing towards -1 does on its least optimal topic (Topic 1). This could be attributed to the overall initial opinions favoring a more positive view, with a slight influence from Topic 3, whose initial opinions are moderate but slightly positive.

Overall Optimal Strategies The presence of intertopic dependencies fundamentally alters the optimal choices for influence maximization. With dependencies considered, Players should make more strategic node selections, favoring ones with higher degrees on topics where the Player is projected to gain the most influence. Relatedly, prioritizing a higher clustering coefficient on topics where projected influence may be lower allows for a more widespread cascade. Finally, for node selection, an individual aligned with the networks' initial opinions on polarizing views is most optimal.

For the message, leveraging indirect influence by deceiving individuals with alignment on a related topic provided better results. Further, it's optimal to polarize the network on topics that already have a strong consensus to indirectly influence those topics that are more moderate or undecided. Additionally, a Player found more influence when using a mixed message strategy.

Results suggest that for effective influence maximization, a Player should

1. Capitalize on disagreement
2. Utilize polarizing messages on topics that already have a polarized consensus
3. Moderate messages on topics that are more neutral or undecided

5.2.2 Two Player

Introducing game-theoretic properties, we assume that Player 1 always sways opinions toward 1– favoring Affirmative Action, supporting Gun Permits, and aligning with the Democratic Party. Conversely, Player 2 always sways opinions toward -1– opposing Affirmative Action, opposing Gun Permits, and aligning with the Republican Party. Conducting 25 simulations across the following two scenarios:

1. **Scenario 1:** Player 1 (Pushing to 1) goes first
2. **Scenario 2:** Player 2 (Pushing to -1) goes first

The following results are obtained:

Table 5.8: Finite Results GSS Data (2 Player, Player 1 Goes First)

Topic	Player 1	Player 2
1: Affirmative Action	**Max Influence:** 82.14 **Optimal Node:** 110 **Message:** [0.30, 0.55, -0.01]	**Max Influence:** 23.68 **Optimal Node:** 126 **Message:**[-0.90, -0.79, -0.71]
2: Gun Permits	**Max Influence:** 22.71 **Optimal Node:** 116 **Message:** [0.00, 1.00, 0.00]	**Max Influence:** 39.66 **Optimal Node:** 70 **Message:** [-0.61, 0.20, 0.06]
3: Political Party	**Max Influence:** 30.11 **Optimal Node:** 94 **Message:** [-0.23, 0.50, 0.91]	**Max Influence:** 33.70 **Optimal Node:** 126 [-0.92, -0.86, -0.73]

Table 5.9: Finite Results GSS Data (2 Player, Player 2 Goes First)

Topic	Player 1	Player 2
1: Affirmative Action	**Max Influence:** 95.71 **Optimal Node:** 2 **Message:**[0.78, 0.69, -0.65]	**Max Influence:** 18.92 **Optimal Node:** 49 **Message:**[-0.83, -0.94, -0.56]
2: Gun Permits	**Max Influence:** 24.91 **Optimal Node:** 176 **Message:**[-0.54, 0.98, 0.82]	**Max Influence:** 80.68 **Optimal Node:** 200 **Message:** [-0.15, 0.12, 0.01]
3: Political Party	**Max Influence:** 65.87 **Optimal Node:** 192 **Message:**[0.35, 0.89, 0.98]	**Max Influence:** 27.40 **Optimal Node:** 278 **Message:** [-0.62, -0.97, -0.81]

Optimal Topic: Similar to what we saw with the 1-Player simulations, Topic 1 was the most optimal for Player 1, and Topic 2 was the most optimal for Player 2. Topic 3, which was initially more moderate but leaning positively, was slightly more optimal for Player 1. This is intuitive because the status quo is moderately aligned with +1, making it easier for Player 1 to influence Topic 3. However, Player 2 gains more influence on Topic 3 when moving second. This indicated that Topic 3 is less polarized and depends more on the strategies of each player.

Maximal Influence In general, the Player who moved second gained more influence over topics. However, Player 2 on Topic 2, which always has more influence, saw more of an impact when moving first. This is attributed to the first-mover advantage[25] where when a Player moves first, they set the initial conditions of the game by pushing an agenda early. When Player 2 moves second, they're forced to respond to the optimal selections of Player 1, which inhibits potential optimal strategies for Player 2. For topics that are polarized or have a clear initial network alignment (i.e. Topic 1 and Topic 2), this advantage gives the first mover more room to create a significant shift in the opinions of the network. This explains the lower influence on Topic 2 for Player 2 when moving second.

Optimal Node: The optimal node varied based on multiple factors, including the initial network opinions, which Player moved first, and the specific topic being targeted.

- For Player 1: When moving first, consistently chose node 116 as the optimal choice for targeting Topic 2. This is primarily because of the node's high clustering coefficient and its alignment with the network's initial consensus on Topic 2, which is polarized in the direction Player 1 is trying to influence. Node 116's initial opinions align closely with Player 1's optimal message for Topic 2, facilitating a wider-spread influence cascade. This consistent choice over all 50 simulations explains why, in response, Player 2 experiences lower influence on Topic 2 compared to when Player 2 moves first.

- For Player 2: When moving second, Player 2 chooses node 126 as optimal for Topic 1 and Topic 3. Node 126 aligns with the network's consensus on Topic 1, which is polarized in Player 2's direction of influence. Further, node 126 has a higher betweenness centrality, indicating that for Topic 3 where initial opinions are moderate, a node with higher betweenness centrality is best to sway individuals toward your view.

Optimal node selection trends allow us to observe that

- When influencing an opinion in the direction of consensus, the optimal node is one that is polarized toward the influence direction. This is especially true if a player moves first.

- For topics where the consensus is moderate or oppositely aligned with your influence direction, choosing a node with higher betweenness centrality is optimal

Optimal Message: Strategies for the optimal message were heavily dependent on the network's initial alignment and polarization on the targeted topic:

- For topics where the network is initially aligned with the Player's view, the optimal strategy is to send a stronger, more polarizing message in the same direction. This approach reinforces the network's existing opinion, making the influence more effective in solidifying the existing consensus. For example, when the network was aligned with 1 on Topic 2, sending a polarizing message in the same direction was optimal

- For topics where the network's initial opinion is unaligned with the Player's influence direction, the optimal strategy involves using indirect influence through the interdependence of topics. This leverages the interconnectedness of topics to shift opinions indirectly toward the Player's desired direction. For example, in the case of Topic 1, where the network was initially aligned with -1, Player 1 used a moderate message to shift the opinion toward 1, while also exploiting the influence of related topics

Analyzing the magnitude of a message for each topic compared to the influence gained in both scenarios corroborates our previous analysis.

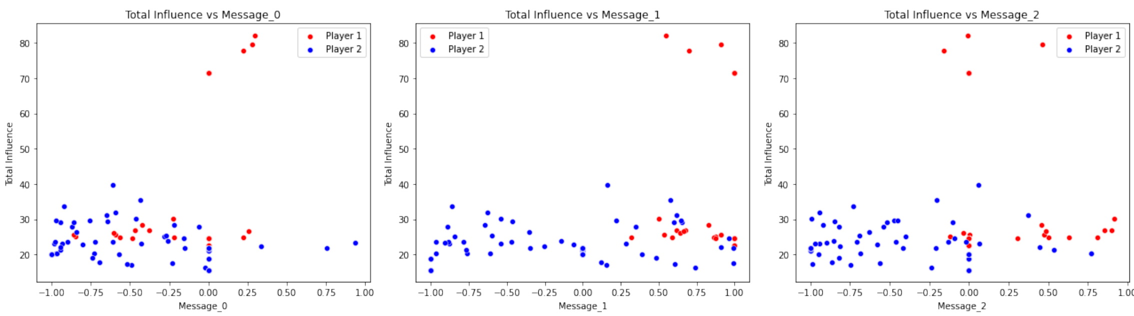


Figure 5.5: Message v Influence (2 Player, Player 1 Goes First)

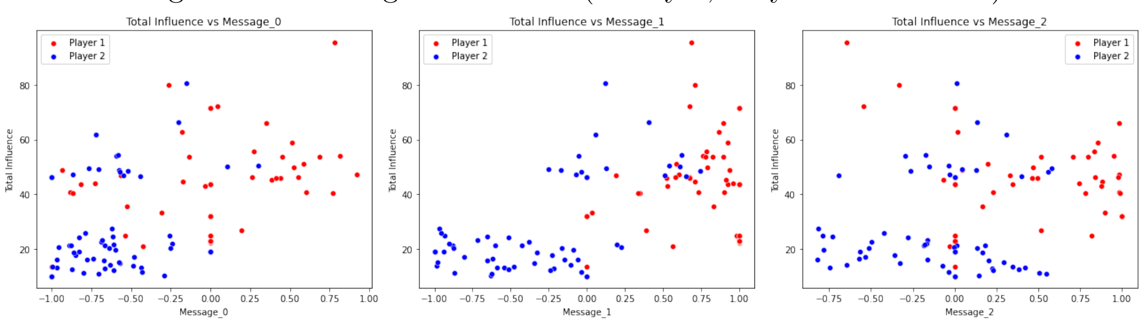


Figure 5.6: Message v Influence (2 Player, Player 2 Goes First)

We parsed the optimal message vector targeting each respective topic for each Player in our 2-Player model. $\texttt{Message}_0$ reflects the message sent on Topic 1 (Affirmative Action), $\texttt{Message}_1$ reflects the message sent on Topic 2 (Gun Permits), $\texttt{Message}_2$ reflects the message sent on Topic 3 (Political Party). We plot the value of the message in the target topics index versus its magnitude of influence.

When Player 1 moves first, the strategies of both players tend to be more stable, likely because the network's initial alignment is slightly positive. This stability suggests that Player 1's move, followed by Player 2's response, results in more predictable and consistent influence strategies.

The highest influence for both players is typically achieved with more moderate messages, particularly for Topic 3. This indicates that, in this scenario, moderate messaging allows both players to capitalize on the existing moderate alignment within the network, resulting in greater overall influence.

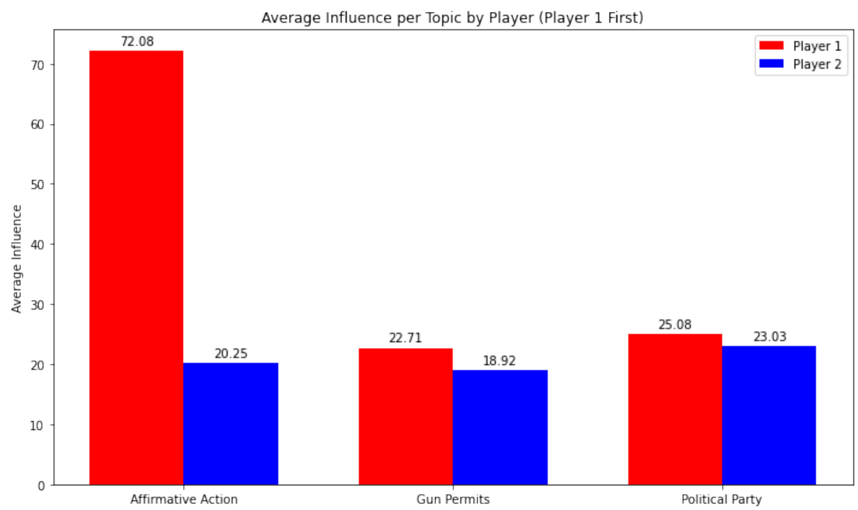


Figure 5.7: Average Influence Over All Topics (2 Player, Player 1 Goes First)

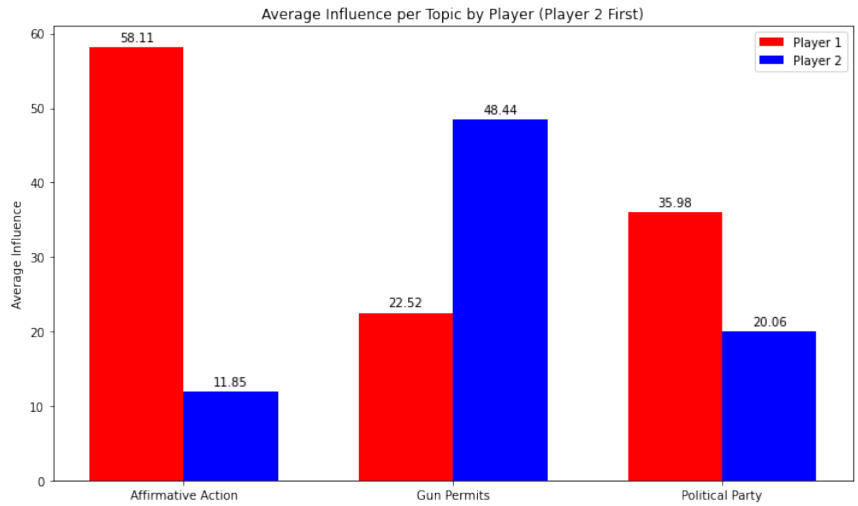


Figure 5.8: Average Influence Over All Topics (2 Player, Player 2 Goes First)

Average influence over all 50 simulations on each topic in each scenario for both Players in our 2-Player model.

Figures 5.7 and 5.8 illustrate that when a player moves first, their influence on the oppositely polarized topic significantly increases. Additionally, they confirm that Player 1 performs better overall, as moving first allows them to maintain a higher average influence on their least optimal topic (Topic 2) compared to Player 2, for whom it is the optimal topic. Furthermore, when Player 2 moves second, they are able to balance the influence on Topic 3. Overall, when a player moves second, they gain influence across all topics except the one with the polar opposite alignment, for which it is most optimal to move first.

Overall Optimal Strategies: Overall, when a player moves second, they gain influence across all topics, except for the one that is most polarized in the opposite direction. For this specific topic, it is most optimal to move first, allowing the first player to shape the network's opinion before the second player can respond. For optimal node selection, Players benefit from nodes with strong alignment with consensus views, especially when influencing polarized topics. To influence moderate views, Players should focus on the nodes metrics, prioritizing higher betweenness centrality, which may lead to a greater cascade of influence. For message, if targeting a topic that the network is initially aligned with the Player's direction of influence, it's optimal for the Player to send a polarizing message in the same direction. For topics where the network's initial opinion is unaligned with the Players' influence direction, it's optimal to use indirect influence via the interdependence of topics.

For all simulations, we collected data on how influence is impacted by the following factors:

1. The targeted topic
2. Optimal node selection
3. Optimal message
4. When the Player moves
5. The network's initial alignment

Based on these results, we analyze the implications for developing optimal strategies for external players aiming to influence a population's opinion in a desired direction. If Bob is the only one influencing Alice and her friends, Bob should send a message that influences a vanilla preference while shifting related opinions, like choices of ice cream toppings. If Alice and her friends already enjoy vanilla, Bob may see limited influence achieved; however should reaffirm their preference and send a strong message favoring vanilla. If Tom steps in, Bob should observe Tom's actions and then respond. Let the majority of Alice and her friends favor caramel sauce, chocolate ice cream, and waffle cones, where their opinion on ice cream flavor and sauce topping is polarized, but the choice of cone is moderate. If preferring caramel sauce is related to preferring vanilla ice cream, Bob should send a message that advocates for caramel sauce, indirectly influencing Alice and her friends to shift towards vanilla. Finally, Bob should select someone who is relatively aligned with liking chocolate ice cream and caramel sauce to initiate his pro-vanilla campaign. In the next section, we synthesize these findings and offer further insights on how to maximize influence.

Chapter 6

Discussion and Conclusions

6.1 Discussion

This paper explored optimal strategies for influence maximization in multidimensional opinion space for single-player and competitive two-player scenarios. With increasing polarization, information overload, and social reliance on technology, this research answered questions regarding how to assert influence dominance, persuading the population towards your view, particularly with existing polarization. Across simulations, we observed consistent patterns revealing the importance of network topology, opinion alignment, message content, and inter-topic dependencies in shaping cascading effects of influence.

For the one-player model, results confirmed the sensitivity of influence outcomes to model parameters. Particularly, the propagation threshold δ governs whether a node forwards information to neighbors with well-aligned opinions. Interestingly, the increase in this propagation threshold has a disproportionally negative impact on total influence compared to the forwarding threshold θ^{high}. This encourages conclusions that the alignment between neighboring nodes plays a vital role in sustaining information cascades than the alignment of nodes and the message vector itself. individual node to maximize global influence. Further, findings from our simulations with synthetic data show that without inertopic dependencies and competition, messages remained relatively polarizing on the targeted topic; this was also verified when using real-world data [20]. Notably, the optimal node for initiating diffusion was not always the one most strongly aligned with the Player's direction of influence, as seen in Tables 8.6 and 8.7. This provides key insight into network structure– when a Player begins diffusion in a node that is already strongly aligned, it amplifies the misalignment with neighbors by intensifying polarization and, therefore, stagnates cascade effects [39][28][30]. Strikingly, the initial alignment of the network had a significant impact on the effectiveness of Players' strategies. When pushing in the opposite direction of the status quo, influence gains were higher– depicting the larger potential for influence when a Player doesn't confirm preexisting views.

For the two-player simulations, introducing intertopic dependencies and game theoretic dynamics [38] provided important complexity. Messages that were most optimal under these conditions tended to be less extreme and showed a preference for a mixed message strategy. This was consistent in our arbitrary and GSS data simulations. The C matrix enabled players to indirectly influence topics by targeting another that was easier to sway depending on the Player's influence direction and the network's initial alignment. This was especially evident when targeting Topic 3 (Political Party), which exhibited a weaker self-influence but exerted a stronger influence on the other topics 3.11. Specifically, Players leveraged the polarizing stances on Topics 1 and 2 to indirectly target and influence Topic 3.

Competition between the two players was further investigated by altering which player moved first. Simulations showed that, generally, the second-mover consistently held an advantage. The second player's ability to observe and react strategically to the fixed strategies of the first mover enabled a more nuanced targeting strategy. However, when the network was misaligned with a Player's stance, moving first was most optimal as it allowed the Player to shape an early narrative. These results indicate a trade-off between initiative and reactivity.

Overall, our research provided significant identification of four core components that remained pivotal for influence maximization:

1. **Targeted Topic**: Polarized topics or ones with higher inter-topic influence are effective leveraging points for influence maximization

2. **Network Alignment**: The initial consensus alignments determine whether a polarizing or moderate message is optimal

3. **Node Selection**: Nodes that are moderately aligned with a message but relatively well aligned on polarized topics and with higher centrality offer optimal cascade potential

4. **Timing of Movement**: In the presence of competition, reactive strategies are optimal except when targeting a topic misaligned with your direction of influence

This collectively emphasizes how optimal strategies for influence are complex, emerging interplays between network topology, topic of interest, and queued position for movement. The research has strong implications for real-world applications regarding political campaigning, public health, and countermeasures for misinformation spread.

6.2 Limitations

A rather obvious limitation to our research is the attempt to simplify complex cognitive processes to mathematical models. Thou we attempt to reflect realistic behavioral and opinion dynamics by enhancing traditional models with parameters like the sociability parameter β, these remain difficult to quantify empirically. For example, while Jafari [24] uses a social skill questionnaire[3] to valuing β, applications may not generalize across contexts. Second, by combining modeling paradigms like FJ[16][15]memory-based updated and linear threshold dynamics [23][24] we inherited trade-offs of convergence characterizations that are captured in existing opinion dynamic models [12][16][15]. Further, our simulations assume that external agents have access to accurate knowledge of the node's opinions. While this may not be entirely realistic, we argue that in real-world contexts, it's becoming increasingly plausible given the oversharing of private information and opinions online and the power of sentiment analysis tools. However, relaxing this assumption would inherit a more realistic approach.

Additional constraints exist from our construction of the network. In the absence of an embedded network in the GSS data, we construct one using probabilistic link predictions founded on homophily frameworks. While this method aligns with sociological principles, its approximation still remains true. Therefore, the application of our model to datasets with embedded network topologies would further confirm the reliability of our insight. Specifically, if the optimal strategies are consistent on data with an embedded network, then that validates the way we've constructed the graph and has exciting applications to represent any data without embedded graphs in a realistic network.

Regarding our data, we further acknowledge the temporal structural limitations of our data. Though the longitudinal panel study offers data from the same individuals over certain years, the GSS[20] captures a two-year gap. The lack of a continuous timescale in the data may under-represent the dynamics of opinion coevolution in our C matrix, which computes the covariant of individuals' opinions over the years.

Finally, from a computational standpoint, we faced significant computational challenges optimizing over both the seed node and multidimensional message vector. To levy this, we define a base message vector with +/-1 in the index of the target topic and add vectors populated with random values from a uniform distribution to restrict our search space. While effective, this heuristic potentially omits globally optimal strategies. Future research should endeavor to find computationally efficient ways to optimize over all possible message vectors.

6.3 Future Research

Our study sets up exciting avenues for extensions in future exploration. With many opinion dynamic models, a primary opportunity lies in empirical validation. Though we've discovered theoretical and simulation-based insights, controlled experiments on individuals may test these dynamics for verification. For example, we encourage running a proctored experiment with individuals where variables are controlled to measure how networks exploded to information interventions influence opinion shifts and cascade mechanisms.

Since a controlled experiment warrants many challenges, machine learning approaches offer promising potential for the forecasting of opinion evolution. Through training predictive models on social media or longitudinal surveys, researchers may approximate group-level responses to targeted messaging. However, because of the stochastic and multivariate nature of human behavior, predictions of this degree offer more accuracy at the group rather than individual level. We encourage future research to look into community structures where it may be more effective and interpretable to target demographic clusters. Specifically, studying whether clustering NY race, gender, income, or political affiliation offers influence pathways that are predictable serves to deepen understanding of homophily[27] and structural advantages in opinion networks.

Another future direction subsists on the dynamics of network structures. While our model assumes an undirected network, meaning connections are binary on an edge, this isn't entirely true, especially in the age of social media. At the next stages of research, incorporating our model to reflect different connections could improve realism. For example, with social media, many one-way interactions, such as follows, likes, or views, are increasingly popular. Incorporating this makes the model more realistic and incorporates additional necessary complexities.

Further, integrating Mean Field Game (MFG) theory could enable the modeling of very large populations as distributions rather than discrete nodes. This serves to represent the unrestricted network connectivity in the presence of social media. A MFG approach to our model offers analytical solutions at the population level strategies of influence, providing parallels to existing work conducted by Bauso [4] and Stella et al.[35].

Importantly, we encourage further work on the ethical dimensions of influence maximization. As influence models become increasingly powerful and applicable, frameworks that ensure responsible use, especially but not constrained to applications in political, public health, or commercial contexts, are extremely critical. Overall, future directions are not limited to the ones presented, as this research has various applicable directions. The future direction discussed above is merely a start to motivate exciting possibilities.

In the end, because Bob read our research and implored strategies discussed, he was able to leverage Alice and her friends' level of connectivity, initial opinions, and strategic movement to persuade them to prefer vanilla ice cream over chocolate.

Chapter 7

Code

For code used in this thesis please refer to the link below.

https://github.com/relsheikh/Thesis.git

Chapter 8

Appendix

8.1 Arbitrary Data

Table 8.1: Optimal Node's Metrics and Initial Opinions 1-Player Arbitrary Data

Topic	**Node**	**Deg.**	**Clust. Coeff.**	**Deg. Centrality**	**Betweenness Centrality**	**Initial Opinion**
1: Affirmative Action	**Push to 1:** 14	10	0.47	0.20	0.0024	[-0.93, 0.82, -0.48]
	Push to -1: 23	15	0.49	0.31	0.0074	[0.97, 0.54, -0.60]
2: Gun Permits	**Push to 1:** 17	13	0.45	0.27	0.0051	[0.55, 0.88, 0.79]
	Push to -1: 46	13	0.63	0.27	0.0029	[-0.27, 0.94, 0.92]
3: Political Party	**Push to 1:** 42	10	0.71	0.20	0.0011	[0.64, 0.72, -0.99]
	Push to -1: 3	13	0.56	0.27	0.0029	[0.42, -0.96, 0.94]

Table 8.2: Optimal Node's Metrics and Initial Opinions 2-Player, Player 1 Goes First Arbitrary Data

Topic	Node	Deg.	Clust. Coeff.	Deg. Centrality	Betweenness Centrality	Initial Opinion
1: Affirmative Action	**Player 1:** 17	13	0.45	0.27	0.0051	[0.55, 0.88, 0.79]
	Player 2: 1	22	0.59	0.45	0.0089	[0.20, -0.69, -0.69]
2: Gun Permits	**Player 1:** 3	13	0.56	0.26	0.0029	[0.42, -0.96, 0.94]
	Player 2: 49	23	0.57	0.47	0.0096	[0.01, -0.90, -0.44]
3: Political Party	**Player 1:** 42	10	0.71	0.20	0.0012	[0.64, 0.72, -0.99]
	Player 2: 3	13	0.56	0.27	0.0029	[0.42, -0.96, 0.94]

Table 8.3: Optimal Node's Metrics and Initial Opinions 2-Player, Player 2 First Arbitrary Data

Topic	Node	Deg.	Clust. Coeff.	Deg. Centrality	Betweenness Centrality	Initial Opinion
1: Affirmative Action	**Player 1:** 11	10	0.42	0.20	0.0029	[0.90, 0.93, 0.62]
	Player 2: 1	22	0.59	0.45	0.0089	[0.20, -0.69, -0.69]
2: Gun Permits	**Player 1:** 17	13	0.45	0.27	0.0051	[0.55, 0.88, 0.79]
	Player 2: 20	27	0.45	0.55	0.0250	[-0.22, -0.46, 0.66]
3: Political Party	**Player 1:** 42	10	0.71	0.20	0.0012	[-0.76, -0.32, 0.89]
	Player 2: 33	20	0.53	0.41	0.0077	[-0.78, -0.94, 0.27]

In Tables 8.1, 8.3, and 8.2, we calculate important node metrics that measure the importance of a node for the optimal nodes in both 1-Player and 2-Player scenarios using the synthetic data. Parameters are initialized still following 3.2

8.2 GSS Data

Table 8.4: Optimal Node's Metrics and Initial Opinion without C

Topic	Node	Deg.	Clust. Coeff.	Deg. Centrality	Betweenness Centrality	Initial Opinion
1: Affirmative Action	**Push to 1:** 70	92	0.53	0.33	0.0023	[-1.00, 1.00, -1.00]
	Push to -1: 126	44	0.32	0.16	0.0016	[0.33, -1.00, 0.67]
2: Gun Permits	**Push to 1:** 276	37	0.36	0.13	0.0010	[1.00, 1.00, 1.00]
	Push to -1: 66	86	0.61	0.31	0.0013	[-1.00, 1.00, -1.00]
3: Political Party	**Push to 1:** 259	115	0.59	0.41	0.0021	[1.00, 1.00, 0.33]
	Push to -1: 70	92	0.53	0.33	0.0023	[-1.00, 1.00, -1.00]

Table 8.5: Optimal Node's Metrics and Initial Opinion w C

Topic	Node	Deg.	Clust. Coeff.	Deg. Centrality	Betweenness Centrality	Initial Opinion
1: Affirmative Action	**Push to 1:** 136	98	0.57	0.35	0.0020	[-1.00, 1.00, -1.00]
	Push to -1: 126	44	0.32	0.16	0.0017	[0.33, -1.00 0.67]
2: Gun Permits	**Push to 1:** 253	19	0.42	0.068	0.00028	[-1.00, 1.00, 0.00]
	Push to -1: 149	88	0.59	0.31	0.0015	[-1.00, 1.00, -1.00]
3: Political Party	**Push to 1:** 106	93	0.60	0.33	0.0015	[-1.00, 1.00, -1.00]
	Push to -1: 200	85	0.61	0.30	0.0013	[-1.00, 1.00, -1.00]

In Tables 8.4 and 8.5, we calculate important node metrics that measure the importance of a node for the optimal nodes in both 1-Player scenarios using the GSS data, with C. Parameters are initialized still following 3.2

Table 8.6: Optimal Node's Metrics and Initial Opinion (2 Player, Player 1 Goes First)

Topic	Node	Deg.	Clust. Coeff.	Deg. Centrality	Betweenness Centrality	Initial Opinion
1: Affirmative Action	**Player 1:** 110	99	0.55	0.35	0.0021	[-1.00, 1.00, -1.00]
	Player 2: 126	44	0.32	0.16	0.0016	[0.33, -1.00, 0.67]
2: Gun Permits	**Player 1:** 116	78	0.55	0.28	0.0013	[1.00, 1.00, 0.67]
	Player 2: 70	92	0.53	0.33	0.0023	[-1.00, 1.00, -1.00]
3: Political Party	**Player 1:** 94	67	0.59	0.24	0.0010	[1.00, 1.00, 1.00]
	Player 2: 126	44	0.32	0.16	0.0016	[0.33, -1.00, 0.67]

Table 8.7: Optimal Node's Metrics and Initial Opinion (2 Player, Player 2 Goes First)

Topic	Node	Deg.	Clust. Coeff.	Deg. Centrality	Betweenness Centrality	Initial Opinion
1: Affirmative Action	**Player 1:** 2	87	0.59	0.31	0.0013	[-1.00, 1.00, -1.00]
	Player 2: 49	36	0.51	0.13	0.0006	[-1.00, -1.00, 1.00]
2: Gun Permits	**Player 1:** 176	67	0.60	0.24	0.0006	[1.00, 1.00, 1.00]
	Player 2: 200	85	0.61	0.30	0.0013	[-1.00, 1.00, -1.00]
3: Political Party	**Player 1:** 192	82	0.63	0.29	0.0012	[-1.00, 1.00, -1.00]
	Player 2: 270	138	0.60	0.49	0.0021	[-0.33, 1.00, -0.33]

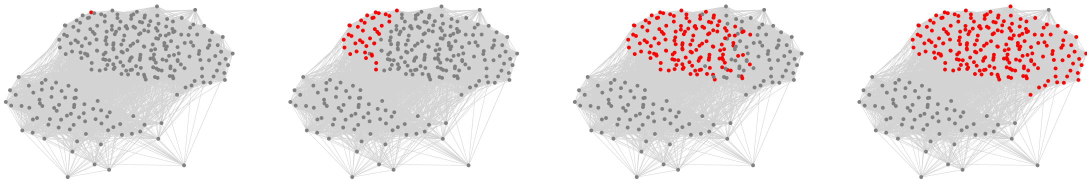
In Tables 8.6 and 8.7 We calculate important node metrics that measure the importance of a node for the optimal nodes in both 2-Player scenarios using the GSS data, with C. Parameters are intialized still following 3.2

Network Simulation Images With GSS Data

1-Player

Topic 1: Affirmative Action

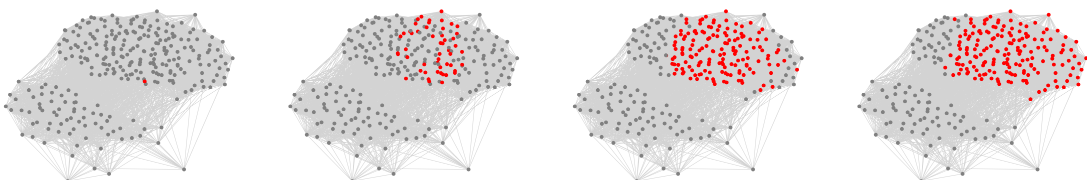
Push to 1, Without C



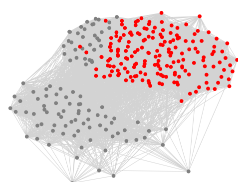
(a) t=1 (b) t=2 (c) t=3 (d) t=4

Figure 8.1: Topic 1 (Affirmative Action) GSS Data Diffusion (1 Player, Push to 1), Without C

Push to 1, With C



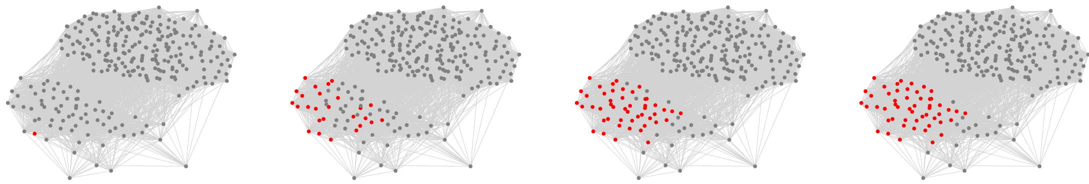
(a) t=1 (b) t=2 (c) t=3 (d) t=4



(e) t=5

Figure 8.2: Topic 1 (Affirmative Action) GSS Data Diffusion (1 Player, Push to 1), With C

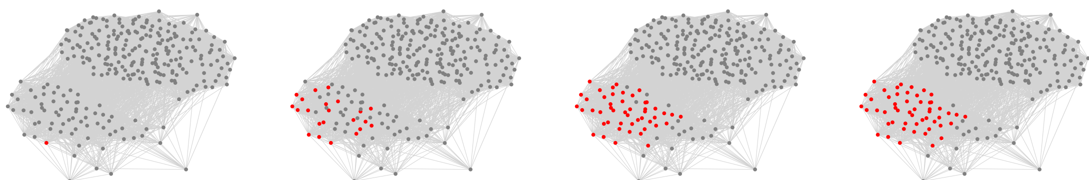
Push to -1, Without C



(a) t=1 (b) t=2 (c) t=3 (d) t=4

Figure 8.3: Topic 1 (Affirmative Action) GSS Data Diffusion (1 Player, Push to -1), Without C

Push to -1, With C

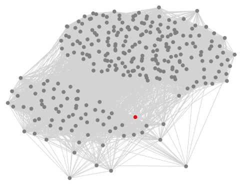


(a) t=1 (b) t=2 (c) t=3 (d) t=4

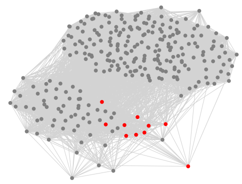
Figure 8.4: Topic 1 (Affirmative Action) GSS Data Diffusion (1 Player, Push to -1), With C

Topic 2: Gun Permits

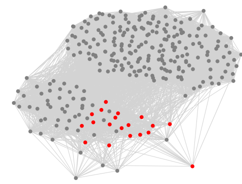
Push to 1, Without C



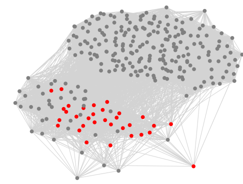
(a) t=1



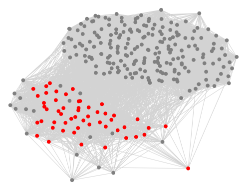
(b) t=2



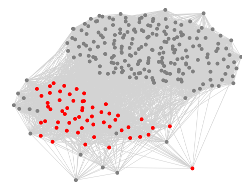
(c) t=3



(d) t=4



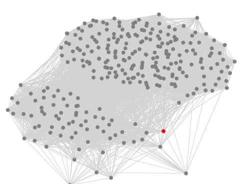
(e) t=5



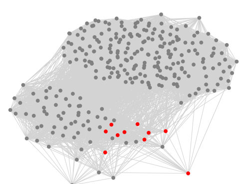
(f) t=6

Figure 8.5: Topic 2 (Gun Permits) GSS Data Diffusion (1 Player, Push to 1), Without C

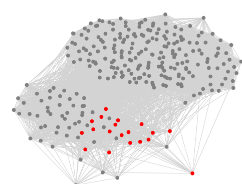
Push to 1, With C



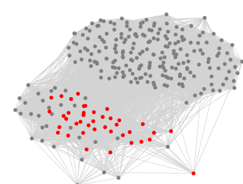
(a) t=1



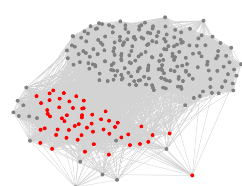
(b) t=2



(c) t=3



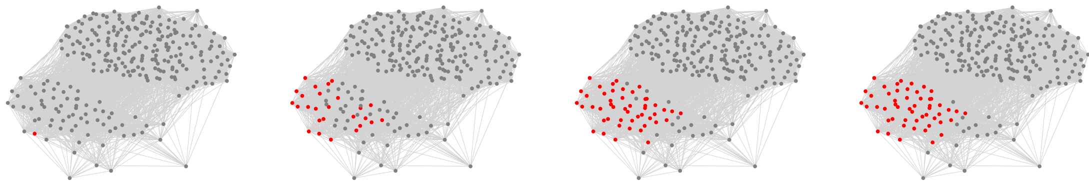
(d) t=4



(e) t=5

Figure 8.6: Topic 2 (Gun Permits) GSS Data Diffusion (1 Player, Push to 1), With C

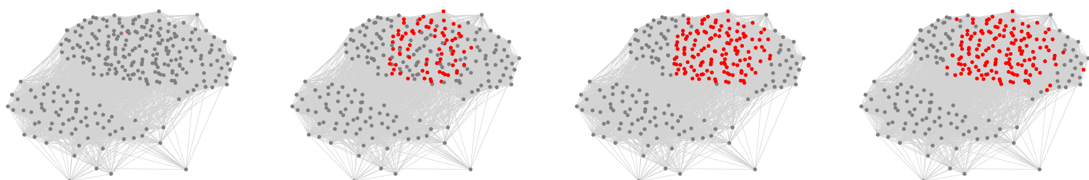
Push to -1, Without C



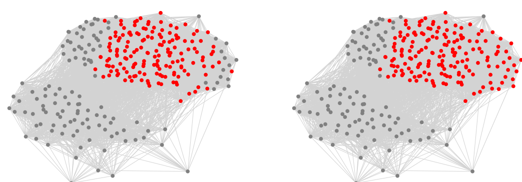
(a) t=1 (b) t=2 (c) t=3 (d) t=4

Figure 8.7: Topic 1 (Affirmative Action) GSS Data Diffusion (1 Player, Push to -1), Without C

Push to -1, With C



(a) t=1 (b) t=2 (c) t=3 (d) t=4

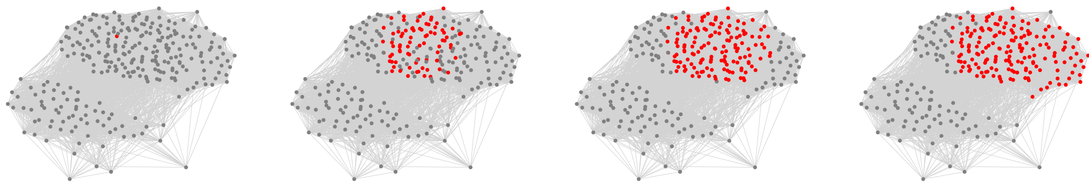


(e) t=5 (f) t=6

Figure 8.8: Topic 2 (Gun Permits) GSS Data Diffusion (1 Player, Push to -1), With C

Topic 3: Political Party

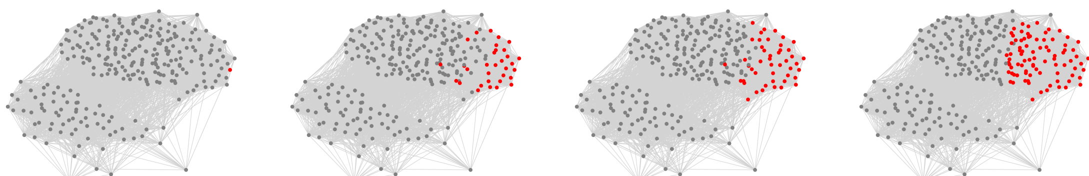
Push to 1, Without C



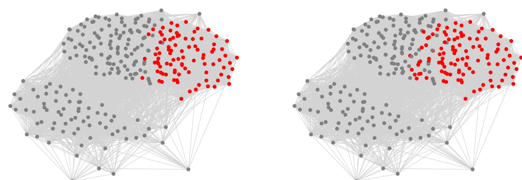
(a) t=1 (b) t=2 (c) t=3 (d) t=4

Figure 8.9: Topic 3 (Political Party) GSS Data Diffusion (1 Player, Push to 1), Without C

Push to 1, With C



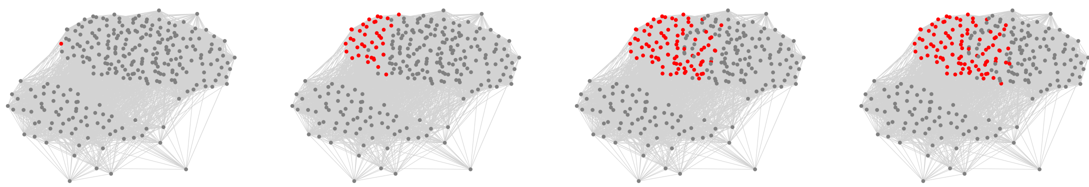
(a) t=1 (b) t=2 (c) t=3 (d) t=4



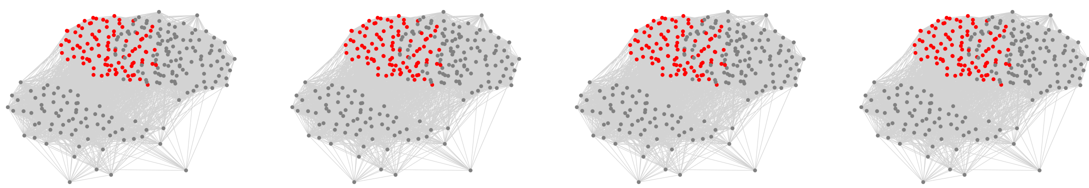
(e) t=5 (f) t=6

Figure 8.10: Topic 3 (Political Party) GSS Data Diffusion (1 Player, Push to 1), With C

Push to -1, Without C



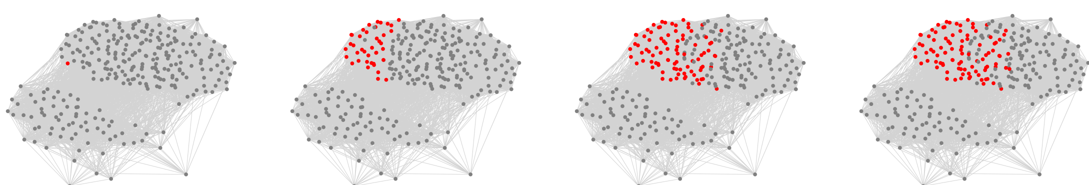
(a) t=1 (b) t=2 (c) t=3 (d) t=4



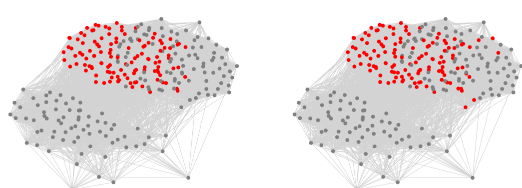
(e) t=5 (f) t=6 (g) t=7 (h) t=8

Figure 8.11: Topic 3 (Political Party) GSS Data Diffusion (1 Player, Push to -1), Without C

Push to -1, With C



(a) t=1 (b) t=2 (c) t=3 (d) t=4



(e) t=5 (f) t=6

Figure 8.12: Topic 3 (Political Party) GSS Data Diffusion (1 Player, Push to -1), With C

2-Player

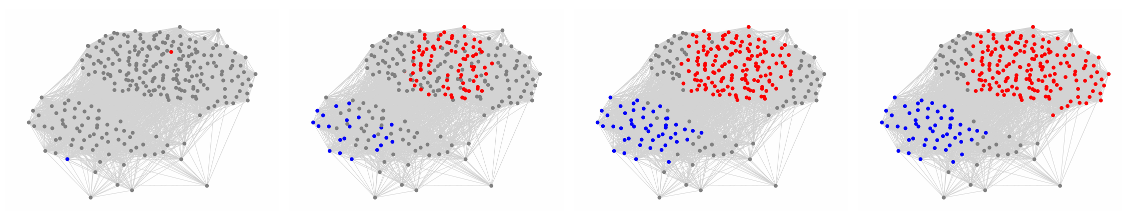
For the visualization below colors correspond to the Players in the following way:

Player 1 (Pushing to +1) → Red

Player 2 (Pushing to -1) → Blue

Topic 1: Affirmative Action

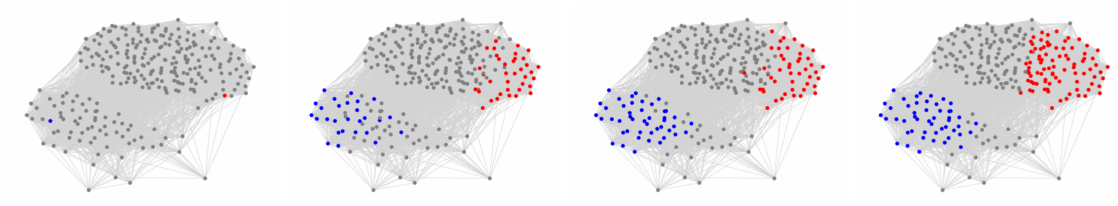
Player 1 First



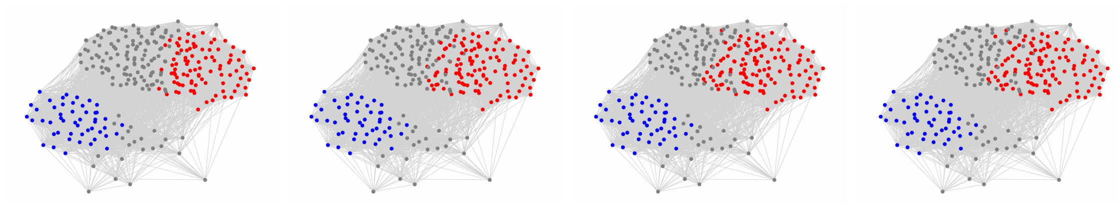
(a) t=1 (b) t=2 (c) t=3 (d) t=4

Figure 8.13: Topic 1 (Affirmative Action) GSS Data Diffusion (2 Player, Player 1 Goes First)

Player 2 First



(a) t=1 (b) t=2 (c) t=3 (d) t=4

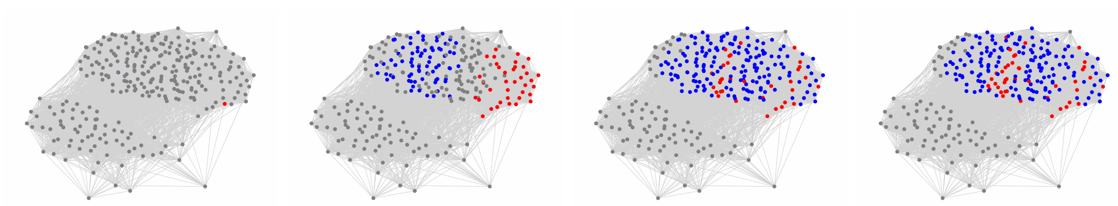


(e) t=5 (f) t=6 (g) t=7 (h) t=8

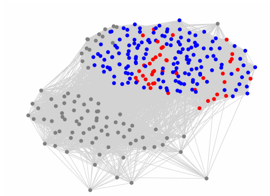
Figure 8.14: Topic 1 (Affirmative Action) GSS Data Diffusion (2 Player, Player 2 Goes First)

Topic 2: Gun Permits

Player 1 First



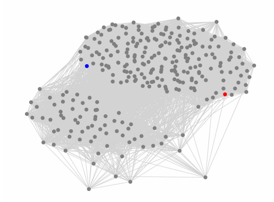
(a) t=1 (b) t=2 (c) t=3 (d) t=4



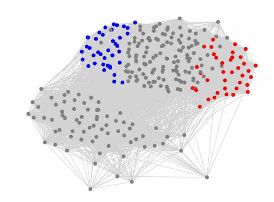
(e) t=5

Figure 8.15: Topic 2 (Gun Permits) GSS Data Diffusion (2 Player, Player 1 Goes First)

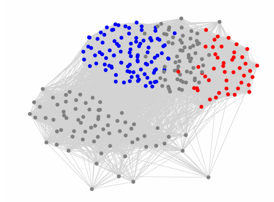
Player 2 First



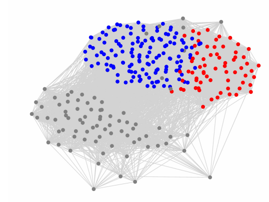
(a) t=1



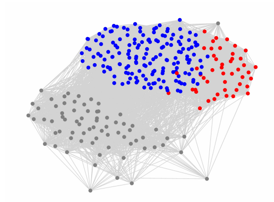
(b) t=2



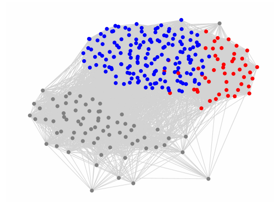
(c) t=3



(d) t=4



(e) t=5

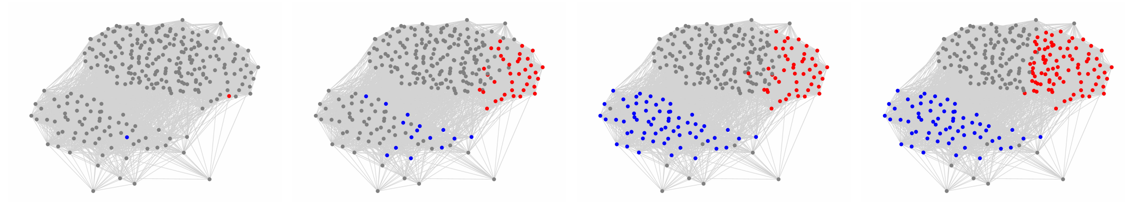


(f) t=6

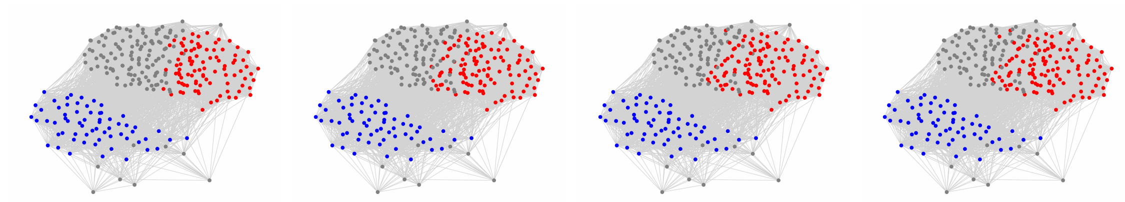
Figure 8.16: Topic 2 (Gun Permits) GSS Data Diffusion (2 Player, Player 2 Goes First)

Topic 3: Political Party

Player 1 First



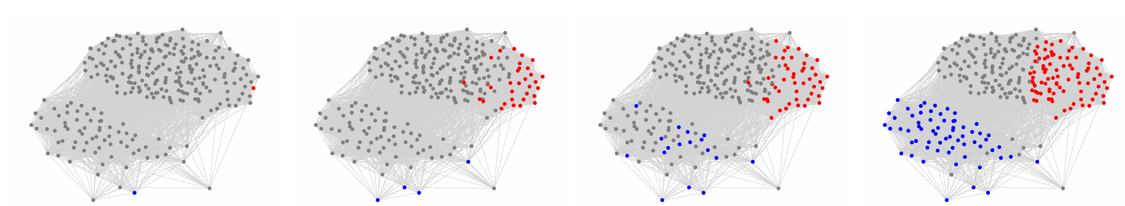
(a) t=1 (b) t=2 (c) t=3 (d) t=4



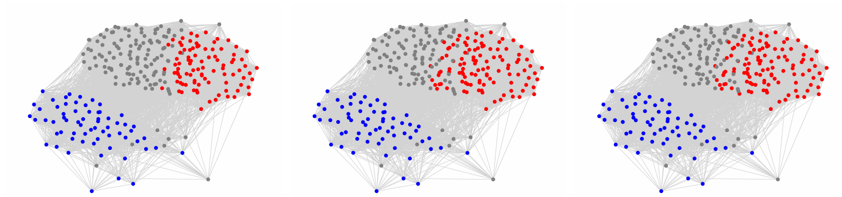
(e) t=5 (f) t=6 (g) t=7 (h) t=8

Figure 8.17: Topic 3 (Political Party) GSS Data Diffusion (2 Player, Player 1 Goes First)

Player 2 First



(a) t=1 (b) t=2 (c) t=3 (d) t=4



(e) t=5 (f) t=6 (g) t=7

Figure 8.18: Topic 3 (Political Party) GSS Data Diffusion (2 Player, Player 2 Goes First)

note ChatGPT was used for debugging and code enhancements

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